Methods to compute ring invariants and applications: a new class of exotic threefolds

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METHODS TO COMPUTE RING INVARIANTS AND APPLICATIONS: A NEW CLASS OF EXOTIC THREEFOLDS

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ABSTRACT. We develop some methods to compute the Makar-Limanov and Derksen invariants, isomorphism classes and automorphism groups for $\mathbf{k}$-domains B, which are constructed from certain Russell $\mathbf{k}$-domains. We propose tools and techniques to distinguish between $\mathbf{k}$-domains with the same Makar-Limanov and Derksen invariants. In particular, we introduce the exponential chain associated to certain modifications. We extract $\mathbb{C}$-domains from the class B that have smooth contractible factorial $\mathrm{Spec}(B)$, which are diffeomorphic to $\mathbb{R}^6$ but not isomorphic to $\mathbb{C}^3$, that is, exotic $\mathbb{C}^3$. We examine associated exponential chains to prove that exotic threefolds $\mathrm{Spec}(B)$ are not isomorphic to $\mathrm{Spec}(R)$, for any Russell $\mathbb{C}$-domain R.

Introduction

This paper discusses some methods to compute Makar-Limanov and Derksen invariants, isomorphism classes and automorphism groups of $\mathbf{k}$-domains. It also proposes some techniques to distinguish between $\mathbf{k}$-domains with the same Makar-Limanov and Derksen invariants.

Let $\mathbf{k}$ be a field of characteristic zero and let A be a commutative $\mathbf{k}$-domain. A $\mathbf{k}$-derivation $\partial \in \mathrm{Der}_{\mathbf{k}}(A)$ is said to be *locally nilpotent* if for every $a \in A$, there is an integer $n \geq 0$ such that $\partial^n(a) = 0$. The *Makar-Limanov invariant* $\mathrm{ML}(A)$ is defined by L. Makar-Limanov as the intersection of the kernels of all locally nilpotent derivations of A. The *Derksen invariant* $\mathcal{D}(A)$ is defined by H. Derksen to be the sub-algebra generated by the kernels of all non-zero locally nilpotent derivations of A. The Makar-Limanov and Derksen invariants are among the more important tools, arising from the study of locally nilpotent derivations, due to their applications in distinguishing between $\mathbf{k}$-domain and in studying isomorphism classes and automorphism groups of $\mathbf{k}$-domain, see e.g. [11, 12, 19, 17, 18, 14, 2, 9].

We improve some techniques used in [1] to compute the Makar-Limanov and Derksen invariants for certain $\mathbf{k}$-domains of the form

$$B \simeq \mathbf{k}[X,Y,Z,T]/\langle X^nY - (Y^m - X^eZ)^d - T^r - X\,Q(X, Y^m - X^eZ, T)\rangle.$$

In [11], S. Kaliman and L. Makar-Limanov developed general techniques to determine the ML-invariant for a class of $\mathbf{k}$-domains $B = \mathbf{k}[X_1,\ldots,X_n]/\mathfrak{b}$. The idea, referred to as the homogeneization technique, is to reduce the problem to the study of homogeneous locally nilpotent derivations on graded algebras $\mathrm{Gr}(B)$ associated to B. For this, one considers suitable filtrations $\mathcal{F} = \{\mathcal{F}_i\}_{i\in\mathbb{R}}$ on B generated by $\mathbb{R}$-weight degree functions ω on $\mathbf{k}[X_1,\ldots,X_n]$, in such a way that every non-zero locally nilpotent derivation on B induces a non-zero homogeneous locally nilpotent derivation on the associated graded algebra $\mathrm{Gr}_{\mathcal{F}}(B)$. The homogeneization technique is efficient when dealing with filtrations that are proper, especially filtrations induced by $\mathbb{R}$-weight degree functions ω, which are appropriate for the ideal $\mathfrak{b}$. Therefore, one surveys weights $\omega(X_i) \in \mathbb{R}$; $i \in \{1,\ldots,n\}$, which guarantee that the ideal $\widehat{\mathfrak{b}}$, generated by top homogenous components of all elements in $\mathfrak{b}$, is prime. We consider a different approach to achieve proper filtrations, that is, we investigate weight degree functions on $\mathbf{k}[X_1,\ldots,X_n,Y_{n+1},\ldots,Y_N] = \mathbf{k}^{[N]}$ for certain choices of $N \in \mathbb{N}$ together with ideals $\mathfrak{a} \subset \mathbf{k}^{[N]}$ such that $B \simeq \mathbf{k}^{[N]}/\mathfrak{a}$ and the ideal $\widehat{\mathfrak{a}}$ is prime.

In a way similar to the one used in [1], we construct the new class from certain Russell $\mathbf{k}$-domains as follows. Given two Russell $\mathbf{k}$-domains $R_i = \mathbf{k}[x,s,t,y_i] \simeq \mathbf{k}[X,Y_i,S,T]/\langle X^{n_i}Y_i - F_i(X,S,T)\rangle$ for $i \in \{1,2\}$, via the localization homomorphism with respect to x, we have $R_1, R_2 \subset \mathbf{k}[x,x^{-1},s,t]$, where $y_i = x^{-n_i}F_i(x,s,t)$. The sub-algebra of $\mathbf{k}[x,x^{-1},s,t]$ generated by R_1 and R_2 coincides with $B := R_1.R_2$ the sub-algebra of $\mathbf{k}[x,x^{-1},s,t]$ consists of all finite sums of elements ab where $a \in R_1$ and $b \in R_2$. That is,

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$B = \mathbf{k}[x,s,t,y_1,y_2] \simeq \mathbf{k}[X,Y_1,Y_2,S,T]/\mathfrak{b}$ for some prime ideal $\mathfrak{b} \subset \mathbf{k}[X,Y_1,Y_2,S,T]$, which clearly contains the ideal $\langle X^{n_1}Y_1 - F_1(X,S,T), X^{n_2}Y_2 - F_2(X,S,T)\rangle$. We show that $\mathcal{D}(B) = \mathbf{k}[x,s,t]$ and $\mathrm{ML}(B) = \mathbf{k}[x]$.

We introduce contraction and exponential chains associated to *exponential modifications*, that is, modifications of $\mathbf{k}$-domains A with locus (a^n, I), where $a \in A$ is an irreducible element, I is an ideal in A and $a^n \in I$. An exponential modification $A[I/a^n]$ has the chain $A[I/a^n] = \langle 1\rangle \supset \langle a\rangle \supset \langle a^2\rangle \supset \cdots \supset \langle a^n\rangle$ of principal ideals in $A[I/a^n]$, which induces the chain $A = \langle 1\rangle^{\mathbf{c}} \supset \langle a\rangle^{\mathbf{c}} \supset \langle a^2\rangle^{\mathbf{c}} \supset \cdots \supset \langle a^n\rangle^{\mathbf{c}}$ of ideals in A, that we call the *contraction chain*, where $\langle a^N\rangle^{\mathbf{c}} = \langle a^N\rangle \cap A$ is the contraction of the ideal $\langle a^N\rangle \subset A[I/a^n]$ with respect to the inclusion $A \hookrightarrow A[I/a^n]$. In turn, the contraction chain give rise to the chain $A \subset A[\langle a\rangle^{\mathbf{c}}/a] \subset A[\langle a^2\rangle^{\mathbf{c}}/a^2] \subset \cdots \subset A[\langle a^n\rangle^{\mathbf{c}}/a^n] = A[I/a^n]$ of sub-algebras of $A[I/a^n]$, which we call the *exponential chain* of $A[I/a^n]$.

In [1], we introduced a family of ring invariants as a generalization of the Derksen invariant. These invariants are certainly useful to distinguish between $\mathbf{k}$-domains with the same Derksen and Makar-Limanov invariants. In this paper we investigate further techniques to distinguish between such $\mathbf{k}$-domains. Certain conditions that two $\mathbf{k}$-domains, with the same Derksen and Makar-Limanov invariants, must verify to be isomorphic can be deduced from properties of their locally nilpotent derivations, see section 4.1. Also, for exponential modifications with the same Derksen and Makar-Limanov invariants, necessary conditions can be given by examining their associated exponential chains, see Proposition 4.11 and Theorem 5.3. Indeed, a $\mathbf{k}$-isomorphism Ψ between exponential modifications $A[I/a^n]$ and R, maps the exponential chain $A \subset A[\langle a\rangle^{\mathbf{c}}/a] \subset A[\langle a^2\rangle^{\mathbf{c}}/a^2] \subset \cdots \subset A[\langle a^n\rangle^{\mathbf{c}}/a^n] = A[I/a^n]$ isomorphically onto $\Psi(A) \subset \Psi(A)[\langle\Psi(a)\rangle^{\mathbf{c}}/\Psi(a)] \subset \cdots \subset \Psi(A)[\langle\Psi(a)^n\rangle^{\mathbf{c}}/\Psi(a)^n] = \Psi(A)[\Psi(I)/\Psi(a)^n] = R$. In particular, if a belongs to the Makar-Limanov invariant and A coincides with the Derksen invariant of the exponential modification $A[I/a^n]$, then the exponential chain is invariant by any $\mathbf{k}$-automorphism of $A[I/a^n]$, and by any locally nilpotent derivation of $A[I/a^n]$. That is, $\psi(A[\langle a^N\rangle^{\mathbf{c}}/a^N]) = A[\langle a^N\rangle^{\mathbf{c}}/a^N]$ and $\partial(A[\langle a^N\rangle^{\mathbf{c}}/a^N]) \subset A[\langle a^N\rangle^{\mathbf{c}}/a^N]$ for every $\mathbf{k}$-automorphism ψ of $A[I/a^n]$, every locally nilpotent derivation ∂ of $A[I/a^n]$ and every $N \in \{1,\ldots,n\}$.

We show that $\mathbf{k}$-domains B of the new class of examples arise as exponential modifications of the Derksen invariant $\mathbf{k}[x,s,t]$ with locus (x^n, I) for certain ideals $I \subset \mathbf{k}[x,s,t]$. Also, we compute the contraction and exponential chains associated to B and we show that the exponential chain characterizes B, then we proceed to determine isomorphism classes and automorphism groups.

In the case $\mathbf{k} = \mathbb{C}$, we extract $\mathbb{C}$-domains from the class B that have smooth contractible factorial $\mathrm{Spec}(B)$, which are diffeomorphic to $\mathbb{R}^6$ but not isomorphic to $\mathbb{C}^3$, as their Makar-Limanov and Derksen invariants are non-trivial, that is, exotic $\mathbb{C}^3$. These new exotic threefolds $\mathrm{Spec}(B)$ are not isomorphic to $\mathrm{Spec}(R)$, for any Russell $\mathbb{C}$-domain R. To show this we compare the associated exponential chains. Indeed, the exponential chain of a $\mathbf{k}$-domain B (of the new class) has some identical members while members of the exponential chain of a Russell $\mathbb{C}$-domains are distinct from (even non-isomorphic to) each other.

1. **Preliminaries**

In this section we briefly recall basic facts in a form appropriate to our needs, see [11, 21]. Unless otherwise specified B will denote a commutative domain over a field $\mathbf{k}$ of characteristic zero. The polynomial ring in n variables over the field $\mathbf{k}$ is denoted by $\mathbf{k}^{[n]}$.

1.1. $\mathbb{Z}$-degree functions, $\mathbb{Z}$-filtrations and associated graded algebras.

Definition 1.1. A $\mathbb{Z}$-*degree function* on B is a map $\deg : B \longrightarrow \mathbb{Z} \cup \{-\infty\}$ such that, for all $a, b \in B$, the following conditions hold:

(1) $\deg(a) = -\infty \Leftrightarrow a = 0$.
(2) $\deg(ab) = \deg(a) + \deg(b)$.
(3) $\deg(a+b) \leq \max\{\deg(a), \deg(b)\}$.

If the equality in (2) is replaced by the inequality $\deg(ab) \leq \deg(a) + \deg(b)$, we say that deg is a $\mathbb{Z}$-*semi-degree function.*

There is a one-to-one correspondence, see e.g. [21, 3], between $\mathbb{Z}$-degree functions and proper $\mathbb{Z}$-filtrations:

Definition 1.2. A $\mathbb{Z}$-*filtration* of B is a collection $\{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ of sub-groups of $(B,+)$ with the following properties:

1- $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ for all $i \in \mathbb{Z}$.
2- $B = \bigcup_{i\in\mathbb{Z}} \mathcal{F}_i$.

3- $\mathcal{F}_i.\mathcal{F}_j \subset \mathcal{F}_{i+j}$ for all $i, j \in \mathbb{Z}$.
The filtration is called *proper* if the following additional properties hold:
4- $\underset{i\in\mathbb{Z}}{\cap}\mathcal{F}_i = \{0\}$.
5- If $a \in \mathcal{F}_i \setminus \mathcal{F}_{i-1}$ and $b \in \mathcal{F}_j \setminus \mathcal{F}_{j-1}$, then $ab \in \mathcal{F}_{i+j} \setminus \mathcal{F}_{i+j-1}$.

Indeed, for a $\mathbb{Z}$-degree function on B, the sub-sets $\mathcal{F}_i = \{b \in B \mid \deg(b) \leq i\}$ are sub-groups of $(B, +)$ that give rise to a proper $\mathbb{Z}$-filtration $\{\mathcal{F}_i\}_{i\in\mathbb{Z}}$. Conversely, every proper $\mathbb{Z}$-filtration $\{\mathcal{F}_i\}_{i\in\mathbb{Z}}$, yields a $\mathbb{Z}$-degree function $\omega : B \longrightarrow \mathbb{Z}\cup\{-\infty\}$ defined by $\omega(0) = -\infty$ and $\omega(b) = i$ if $b \in \mathcal{F}_i \setminus \mathcal{F}_{i-1}$, such an integer i exists by property 4 of proper filtrations.

Definition 1.3. Given a $\mathbf{k}$-domain $B = \underset{i\in\mathbb{Z}}{\cup}\mathcal{F}_i$ equipped with a proper $\mathbb{Z}$-filtration $\mathcal{F} = \{\mathcal{F}_i\}_{i\in\mathbb{Z}}$, the associated graded algebra $\mathrm{Gr}(B)$ is the abelian group

$$\mathrm{Gr}(B) := \underset{i\in\mathbb{Z}}{\oplus}\mathcal{F}_i/\mathcal{F}_{i-1}$$

equipped with the unique multiplicative structure for which the product of the elements $a+\mathcal{F}_{i-1} \in \mathcal{F}_i/\mathcal{F}_{i-1}$ and $b+\mathcal{F}_{j-1} \in \mathcal{F}_j/\mathcal{F}_{j-1}$, where $a \in \mathcal{F}_i$ and $b \in \mathcal{F}_j$, is the element

$$(a+\mathcal{F}_{i-1})(b+\mathcal{F}_{j-1}) := ab+\mathcal{F}_{i+j-1} \in \mathcal{F}_{i+j}/\mathcal{F}_{i+j-1}.$$

Property 5 for a proper filtration in Definition 1.2 ensures that $\mathrm{Gr}(B)$ is a commutative $\mathbf{k}$-domain when B is an integral domain. Since for each $a \in B \setminus \{0\}$ the set $\{n \in \mathbb{Z} \mid a \in \mathcal{F}_n\}$ has a minimum (by property 4 of proper filtrations), there exists i such that $a \in \mathcal{F}_i$ and $a \notin \mathcal{F}_{i-1}$. So we can define a $\mathbf{k}$-linear map $\mathrm{gr} : B \longrightarrow \mathrm{Gr}(B)$ by sending a to its class in $\mathcal{F}_i/\mathcal{F}_{i-1}$, i.e $b \mapsto b+\mathcal{F}_{i-1}$, and $\mathrm{gr}(0) = 0$. We will frequently denote $\mathrm{gr}(b)$ simply by $\hat{b}$. Observe that $\mathrm{gr}(b) = 0$ if and only if $a = 0$.

1.2. **The homogeneization technique.**

Definition 1.4. By a $\mathbf{k}$-*derivation* of B, we mean a $\mathbf{k}$-linear map $D : B \longrightarrow B$ which satisfies the Leibniz rule: For all $a, b \in B$; $D(ab) = aD(b) + bD(a)$. The *kernel* of a derivation D is the subalgebra $\ker D = \{b \in B; D(b) = 0\}$ of B. A $\mathbf{k}$-derivation $D \in \mathrm{Der}_{\mathbf{k}}(B)$ is said to be *locally nilpotent* if for every $a \in B$, there exists $n \in \mathbb{Z}_{\geq 0}$ (depending of a) such that $\partial^n(a) = 0$. The set of all locally nilpotent derivations of B is denoted by $\mathrm{LND}(B)$.

It is convenient to reduce the study of $\mathrm{LND}(B)$ to the study of homogeneous locally nilpotent derivations on a graded algebra $\mathrm{Gr}_{\mathcal{F}}(B)$, associated to a suitable filtration $\mathcal{F} = \{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ of B, in such a way that every non-zero locally nilpotent derivation on B induces a non-zero homogeneous locally nilpotent derivation on the associated graded algebra $\mathrm{Gr}_{\mathcal{F}}(B)$. This technique, which is due to Makar-Limanov [18], of replacing a locally nilpotent derivation by the induced homogeneous one is called "homogeneization of derivations" or simply *homogeneization* technique, see [5].

Definition 1.5. Given a $\mathbf{k}$-domain $B = \underset{i\in\mathbb{Z}}{\cup}\mathcal{F}_i$ equipped with a proper $\mathbb{Z}$-filtration, a $\mathbf{k}$-derivation D of B is said to *respect* the filtration if there exists an integer τ such that $D(\mathcal{F}_i) \subset \mathcal{F}_{i+\tau}$ for all $i \in \mathbb{Z}$. The smallest integer τ, such that $D(\mathcal{F}_i) \subset \mathcal{F}_{i+\tau}$ for all $i \in \mathbb{Z}$, is called the *degree* of D with respect to $\mathcal{F} = \{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ and denoted by $\deg_{\mathcal{F}} D$.
Note that if D respects the filtration $\mathcal{F} = \{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ then $\deg_{\mathcal{F}} D$ is well-defined. Indeed, denote by deg the $\mathbb{Z}$-degree function corresponding to $\mathcal{F} = \{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ and let U be the non-empty subset of $\mathbb{Z}\cup\{-\infty\}$ defined by $U := \{\deg(D(b)) - \deg(b)\ ;\ b \in B \setminus \{0\}\}$. Since D respects the filtration $\mathcal{F}$, the set U is bounded above by τ_0. Thus it has a greatest element τ which coincides with $\deg_{\mathcal{F}} D$ by definition.
Suppose that D respects the filtration $\mathcal{F} = \{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ and let $\tau = \deg_{\mathcal{F}} D$. We define a $\mathbf{k}$-linear map $\widehat{D}$: $\mathrm{Gr}(B) \longrightarrow \mathrm{Gr}(B)$ as follows: If $D = 0$, then $\widehat{D} = 0$ the zero map. Otherwise, if $D \neq 0$ then we define

$$\widehat{D} : \mathcal{F}_i/\mathcal{F}_{i-1} \longrightarrow \mathcal{F}_{i+\tau}/\mathcal{F}_{i+\tau-1}$$

by the rule $\widehat{D}(a+\mathcal{F}_{i-1}) = D(a)+\mathcal{F}_{i+\tau-1}$. Now extend $\widehat{D}$ to all of $\mathrm{Gr}(B)$ by linearity. One checks that $\widehat{D}$ satisfies the Leibniz rule, therefore it is a homogeneous $\mathbf{k}$-derivation of $\mathrm{Gr}(B)$ of degree τ, that is, $\widehat{D}$ sends homogeneous elements of degree i to either the zero element in $\mathrm{Gr}(B)$ or to homogeneous elements of degree $i+\tau$. Note that $\widehat{D} = 0$ if and only if $D = 0$, and that $\mathrm{gr}(\ker D) \subset \ker \widehat{D}$.

1.3. **$\mathbb{Z}$-weight degree functions.**
Let $\mathfrak{b}$ be a prime ideal in $\mathbf{k}^{[n]}$, in this paper we are interested in $\mathbb{Z}$-degree functions deg on $\mathbf{k}^{[n]}/\mathfrak{b}$, which are induced by $\mathbb{Z}$-weight degree functions on the polynomial algebra $\mathbf{k}^{[n]}$. Degree functions deg that satisfy $\deg(\lambda) = 0$ for every $\lambda \in \mathbf{k}\backslash\{0\}$ is referred to as degree functions *over* $\mathbf{k}$.

Definition 1.6. A $\mathbb{Z}$-*weight degree function* on the polynomial algebra $\mathbf{k}^{[n]} = \mathbf{k}[X_1, \dots, X_n]$ is a $\mathbb{Z}$-degree function (over $\mathbf{k}$) ω such that $\omega(P) = \max\{\omega(M)\,;\, M \in \mathcal{M}(P)\}$, where $P \in \mathbf{k}^{[n]}$ is a non-zero polynomial, and $\mathcal{M}(P)$ is the set of non-zero monomials of P. Clearly, ω is uniquely determined by the weights $\omega(X_i) \in \mathbb{Z}$, $i \in \{1, \dots, n\}$. A $\mathbb{Z}$-weight degree function ω defines a grading $\mathbf{k}^{[n]} = \oplus_{l\in\mathbb{Z}}\mathbf{k}_l^{[n]}$ where $\mathbf{k}_l^{[n]} \setminus \{0\}$ consists of all the ω-homogeneous polynomials of ω-degree l. Accordingly, for any $P \in \mathbf{k}^{[n]}\backslash\{0\}$ we have a unique decomposition $P = P_{l_1} + \cdots + P_{l_j}$ into a sum of ω-homogeneous components P_{l_i} of ω-degree l_i where $l_1 < l_2 < \cdots < l_j = \omega(P)$. We call $\widehat{P} := P_{l_j}$ the *highest homogeneous component* of P or the *principal component* of P. It is clear that $\widehat{PQ} = \widehat{P}\widehat{Q}$.

Given a finitely generated $\mathbf{k}$-domain $B \simeq \mathbf{k}^{[n]}/\mathfrak{b}$ where $\mathfrak{b}$ is a prime ideal in $\mathbf{k}^{[n]}$, let $\pi : \mathbf{k}^{[n]} \longrightarrow B$ be the natural projection. Denote by $\widehat{\mathfrak{b}}$ the (graded) ideal in $\mathbf{k}^{[n]}$ generated by the highest homogeneous components of all elements of $\mathfrak{b}$.

Definition 1.7. We say that a $\mathbb{Z}$-weight degree function ω on $\mathbf{k}^{[n]}$ is *appropriate for an ideal* $\mathfrak{b}$ if the following conditions hold:
(a) $\mathfrak{b} \subset \langle X_1, \dots, X_n\rangle$.
(b) The ideal $\widehat{\mathfrak{b}}$ is prime and $X_i \notin \widehat{\mathfrak{b}}$; $\forall i = 1, \dots, n$.
Assume that ω on $\mathbf{k}^{[n]}$ is appropriate for the ideal $\mathfrak{b}$, for every non-zero $p \in B$ set

$$\omega_B(p) := \min_{P\in\pi^{-1}(p)} \omega(P).$$

The next Proposition 1.8, which is due to Kaliman and Makar-Limanov, ensures that ω_B is a $\mathbb{Z}$-degree function on B. Therefore, the filtration $\mathcal{F}_{\omega_B} = \{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ induced by ω_B is a proper $\mathbb{Z}$-filtration of $B \simeq \mathbf{k}[X_1, \dots, X_n]/\mathfrak{b}$. Moreover, the proposition provides a description of the associated graded algebra $\mathrm{Gr}(B)$. Finally, it asserts in particular that every locally nilpotent derivation respects the proper filtration $\mathcal{F}_{\omega_B}$.

Proposition 1.8. [11, Lemma 3.2, Proposition 4.1, and Lemma 5.1] *Let $B = \mathbf{k}[x_1, \dots, x_n] \simeq \mathbf{k}[X_1, \dots, X_n]/\mathfrak{b}$ be a finitely generated $\mathbf{k}$-domain and let ω be a $\mathbb{Z}$-weight degree function on $\mathbf{k}[X_1, \dots, X_n]$. Suppose that ω is appropriate for the ideal $\mathfrak{b}$, then:*
(1) *ω_B is a $\mathbb{Z}$-degree function on B and $\omega_B(x_i) = \omega(X_i)\,;\, i = 1, \dots, n$.*
(2) *The graded algebra $\mathrm{Gr}(B)$ associated to the proper $\mathbb{Z}$-filtration $\mathcal{F}_{\omega_B} = \{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ is isomorphic to $\mathbf{k}^{[n]}/\widehat{\mathfrak{b}}$.*
(3) *Every derivation ∂ of B respects the ω_B-filtration $\mathcal{F}_{\omega_B} = \{\mathcal{F}_i\}_{i\in\mathbb{Z}}$, that is, there exists τ such that $\partial(F_i) \subset F_{i+\tau}$ for every $i \in \mathbb{Z}$. Consequently, $\deg_{\omega_B}(\partial) < \infty$ and ∂ induces a derivation $\widehat{\partial}$ of $\mathrm{Gr}(B)$ which is locally nilpotent whenever ∂ is.*

2. **A New Class of Examples**

In this section, we consider a family of commutative finitely generated $\mathbf{k}$-domains of the following form:

$$B := \mathbf{k}[x, y, z, t] \simeq \mathbf{k}[X, Y, Z, T]/\langle X^nY - (Y^m - X^eZ)^d - T^r - X\,Q(X, Y^m - X^eZ, T)\rangle,$$

where $e \geq 0$, $n \geq 1$ such that $(n, e) \neq (1, 0)$, $m, d, r \geq 2$ such that $\gcd(d, r) = 1$, and $Q(X, S, T) \in \mathbf{k}[X, S, T]$.

2.1. **Algebraic construction.**
Here, we explain how to construct the new class B from Russell $\mathbf{k}$-domains:

Definition 2.1. Given an integer $n \in \mathbb{N}$ and a polynomial $F(X, S, T) \in \mathbf{k}[X, S, T]$ such that $P(S, T) := F(0, S, T) \notin \mathbf{k}$, we define the *Russell* $\mathbf{k}$-domain corresponding to the pair (n, F) to be the $\mathbf{k}$-domain;

$$\mathbf{R}_{(n,F)} := \mathbf{k}[x, y, s, t] \simeq \mathbf{k}[X, Y, S, T]/\langle X^nY - F(X, S, T)\rangle.$$

Consider the Russell $\mathbf{k}$-domain $R = \mathbf{R}_{(n,S^d+T^r+X\,Q(X,S,T))}$ corresponding to the pair $(n, S^d + T^r + X\,Q(X, S, T))$. It is isomorphic to $\mathbf{k}[X, Y, Z, T]/\langle X^nY - (Y^m - Z)^d - T^r - X\,Q(X, Y^m - Z, T)\rangle$, which is a member of the new family 2 that corresponds to $e = 0$.

Denote by z the element $z := x^{-nm-e}((s^d + t^r + x\,Q(x,s,t))^m - x^{nm}s) \in \mathbf{k}[x, x^{-1}, s, t]$. That is, z is an algebraic element over $\mathbf{k}[x,s,t]$ that has the following minimal polynomial;

$$x^{nm+e}Z - (s^d + t^r + x\,Q(x,s,t))^m + x^{nm}s \in \mathbf{k}[x,s,t][Z].$$

Thus,

$$\mathbf{k}[x,s,t,z] \simeq \mathbf{k}[X,S,T,Z]/\langle X^{nm+e}Z - (S^d + T^r + X\,Q(X,S,T))^m + X^{nm}S\rangle.$$

The ring $\mathbf{k}[x,s,t,z]$ is the Russell $\mathbf{k}$-domain corresponding to the pair $(nm+e, (S^d + T^r + X\,Q(X,S,T))^m - X^{nm}S)$.

Extend R to $B := R[z]$, the sub-algebra of $\mathbf{k}[x, x^{-1}, s, t]$ generated by z and $R \subset \mathbf{k}[x, x^{-1}, s, t]$, where the inclusion is induced be the localization homomorphism with respect to x. Then,

$$B = \mathbf{k}[x,y,s,t,z] \simeq \mathbf{k}[X,Y,Z,S,T]/\langle X^nY - S^d - T^r - X\,Q(X,S,T), Y^m - X^eZ - S\rangle.$$

Hence,

$$B \simeq \mathbf{k}[X,Y,Z,T]/\langle X^nY - (Y^m - X^eZ)^d - T^r - X\,Q(X, Y^m - X^eZ, T)\rangle.$$

Note that $R, \mathbf{k}[x,s,t,z] \subset B = R.\mathbf{k}[x,s,t,z]$, where $R.\mathbf{k}[x,s,t,z]$ is by definition the sub-algebra of $\mathbf{k}[x, x^{-1}, s, t]$ consists of all finite sums of elements ab where $a \in R$ and $b \in \mathbf{k}[x,s,t,z]$. This simply means that B can be realized as the sub-algebra of $\mathbf{k}[x, x^{-1}, s, t]$ generated by both R and $\mathbf{k}[x,s,t,z]$.

2.2. $\mathbb{Z}$-degree functions, $\mathbb{Z}$-filtrations, and associated graded algebras.

Given a $\mathbf{k}$-domain $A \simeq \mathbf{k}[X_1, \ldots, X_n]/\mathfrak{a}$, we consider proper $\mathbb{Z}$-filtrations on A induced by $\mathbb{Z}$-weight degree functions on $\mathbf{k}[X_1, \ldots, X_n, Y_{n+1}, \ldots, Y_N] = \mathbf{k}^{[N]}$ for certain choices of $N \in \mathbb{N}$ together with ideals $\mathfrak{b} \subset \mathbf{k}^{[N]}$ such that the ring A can be identified with $\mathbf{k}^{[N]}/\mathfrak{b}$ and the ideal $\widehat{\mathfrak{b}}$ is prime. We refer to this technique as the twisted embedding technique, see [1, Sub-section 2.2.2]. It is also convenient to apply the homogeneization technique to proper filtrations $\{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ which give raise to graded algebras with one dimensional graded pieces, that is, the corresponding graded pieces $A_{[i]} := \mathcal{F}_i/\mathcal{F}_{i-1}$ are generated by one element as $A_{[0]}$-modules. In particular, this is the case for filtrations $\{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ that satisfy the condition: for every $i \in \mathbb{Z}$, the $\mathcal{F}_0$-module $\mathcal{F}_i$ is generated by $|i| + 1$ element.

Note that the $\mathbf{k}$-domain

$$B := \mathbf{k}[x,y,z,t] \simeq \mathbf{k}[X,Y,Z,T]/\langle X^nY - (Y^m - X^eZ)^d - T^r - X\,Q(X, Y^m - X^eZ, T)\rangle$$

is isomorphic to $\mathbf{k}[X,Y,Z,S,T]/\mathfrak{b}$, where $\mathfrak{b}$ the ideal in $\mathbf{k}^{[5]} = \mathbf{k}[X,Y,Z,S,T]$ defined by

$$\mathfrak{b} = \langle X^nY - S^d - T^r - X\,Q(X,S,T),\ Y^m - X^eZ - S\rangle.$$

That is, $B = \mathbf{k}[x,y,z,t] = \mathbf{k}[x,s,t,y,z] \simeq \mathbf{k}^{[5]}/\mathfrak{b}$, where $s = y^m - x^ez$.

Definition 2.2. Let ω be the $\mathbb{Z}$-weight degree function on $\mathbf{k}^{[5]}$ defined by

$$\omega(X,Y,Z,S,T) = (-1, n, nm+e, 0, 0).$$

Let $\widehat{\mathfrak{b}}$ be the ideal in $\mathbf{k}^{[5]}$ generated by highest homogeneous components, relative to ω, of all elements in $\mathfrak{b}$. The highest homogeneous components of $X^nY - S^d - T^r - X\,Q(X,S,T)$ and $Y^m - X^eZ - S$ are the irreducible polynomials $X^nY - S^d - T^r$ and $Y^m - X^eZ$ (respectively) in $\mathbf{k}^{[5]}$. Using properties of the graded map $\mathrm{gr}_\omega : \mathbf{k}^{[5]} \longrightarrow \mathbf{k}^{[5]}$ presented in [1, Lemma 1.4], one checks that the ideal $\widehat{\mathfrak{b}}$ coincides with $\langle X^nY - S^d - T^r,\ Y^m - X^eZ\rangle$, see also Remark 2.3 below. Furthermore, the ideal $\widehat{\mathfrak{b}} = \langle X^nY - S^d - T^r,\ Y^m - X^eZ\rangle$ is prime. Indeed, note that $\mathbf{k}^{[5]}/\widehat{\mathfrak{b}} \simeq \mathbf{R}_{(n,S^d+T^r)}[Z]/\langle y^m - x^eZ\rangle$, where $\mathbf{R}_{(n,S^d+T^r)} = \mathbf{k}[x,s,t,y] \simeq \mathbf{k}[X,Y,S,T]/\langle X^nY - S^d - T^r\rangle$ is the Russell $\mathbf{k}$-domain corresponding to the pair $(n, S^d + T^r)$. Since a polynomial of degree one $P(Z) = aZ + b \in \mathbf{R}_{(n,S^d+T^r)}[Z]$ is irreducible if and only if a and b have no common factors in $\mathbf{R}_{(n,S^d+T^r)}$, we conclude that $y^m - x^eZ \in \mathbf{R}_{(n,S^d+T^r)}[Z]$ is irreducible. Furthermore, by Gauss's Lemma, $y^m - x^eZ$ is prime as $\mathbf{R}_{(n,S^d+T^r)}[Z]$ is factorial (since $\mathbf{R}_{(n,S^d+T^r)}$ is factorial by virtue of [20, Lemma 1]). Therefore, we deduce that $\mathbf{R}_{(n,S^d+T^r)}[Z]/\langle y^m - x^eZ\rangle \simeq \mathbf{k}^{[5]}/\widehat{\mathfrak{b}}$ is a $\mathbf{k}$-domain and hence $\widehat{\mathfrak{b}}$ is prime.

Remark 2.3. Let $\mathfrak{a} = \langle P, Q\rangle$ be the ideal (not necessary prime) generated by elements $P, Q \in \mathbf{k}^{[N]}$, and let ω be a weight degree on $\mathbf{k}^{[N]}$. Recently Moser-Jauslin informed us that $\widehat{\mathfrak{a}} = \langle \hat{P}, \hat{Q}\rangle$ whenever $\gcd(\hat{P}, \hat{Q}) = 1$ and provided the following argument. Given $H \in \mathfrak{b}$ there exist $f, g \in \mathbf{k}^{[N]}$ such that $H = fP + gQ$. Note that the pair (f,g) can be chosen such that $\omega(fP) \leq \omega(H)$. Indeed, if not then for every such pair (f,g) we have

$\omega(fP), \omega(gQ) > \omega(H)$. Thus $\omega(f)$ bounded below by $\omega(H) - \omega(P)$. So f can be chosen to be of minimal degree. On the other hand, condition $\omega(fP), \omega(gQ) > \omega(H)$ implies that $\omega(fP) = \omega(gQ)$ and $\hat{f}\hat{P} + \hat{g}\hat{Q} = 0$, see [1, Lemma 1.4 (4)]. Since $\gcd(\hat{P}, \hat{Q}) = 1$, we conclude $\hat{Q}$ divides $\hat{f}$. Write $\hat{f} = u\hat{Q}$ and let $f_0 = f - uQ$. Then we get $H = f_0 P + (g + uP)Q$ with $\omega(f_0) < \omega(f)$. This contradicts the minimality of the degree of f. Therefore, since $\omega(fP) \leq \omega(H)$, we conclude that $\hat{H}$ is either $\hat{g}\hat{Q}$ (if $\omega(fP) < \omega(gQ)$), see [1, Lemma 1.4 (2)], or $\hat{f}\hat{P} + \hat{g}\hat{Q}$ (if $\omega(fP) = \omega(gQ)$), see [1, Lemma 1.4 (3)]. Hence, $\widehat{\mathfrak{a}} = \langle \hat{P}, \hat{Q} \rangle$.

Thus, we conclude that ω is appropriate for the ideal $\mathfrak{b}$ and hence ω induces ω_B a $\mathbb{Z}$-degree function on B, see Proposition 1.8 (1), where

$$\omega_B(p) := \min_{P \in \pi^{-1}(p)} \{\omega(P)\}.$$

Noting that the proper $\mathbb{Z}$-filtration of $\mathbf{k}[X, Y, Z, S, T]$ induced by ω is given by

$$\Im_\alpha = \bigoplus_{\alpha = (nm+e)i + nj - l} \mathbf{k}[S, T].X^l Y^j Z^i \oplus \Im_{\alpha - 1}\,; \;\; i, j, l \in \mathbb{Z},$$

we obtain the following.

Proposition 2.4. *Let $\mathcal{F} = \{\mathcal{F}_\alpha = \pi(\Im_\alpha)\}_{\alpha \in \mathbb{Z}}$ be the proper $\mathbb{Z}$-filtration on B induced by ω_B. Then:*
(1) $\mathcal{F}_{-i} = \mathbf{k}[s, t]x^i + \mathcal{F}_{-i-1}$ *for every* $i > 0$,
(2) $\mathcal{F}_0 = \mathbf{k}[s, t] + \mathcal{F}_{-1} = \mathbf{k}[x, s, t]$,
(3) $\mathcal{F}_{nj-l} = \mathbf{k}[s, t]x^l y^j + \mathcal{F}_{nj-l-1}$ *for* $l \in \{0, \ldots, n-1\}$ *and* $j \in \mathbb{N}$,
(4) $\mathcal{F}_{(nm+e)i-l} = \mathbf{k}[s, t]x^l z^i + \mathcal{F}_{(nm+e)i-l-1}$ *for* $l \in \{0, \ldots, e-1\}$ *and* $i \in \mathbb{N}$,
(5) $\mathcal{F}_{(nm+e)i+nj-l} = \mathbf{k}[s, t]x^l y^j z^i + \mathcal{F}_{(nm+e)i+nj-l-1}$ *for* $l \in \{0, \ldots, \min\{n, e\} - 1\}$ *and* $i, j \in \mathbb{N}\backslash\{0\}$.

Corollary 2.5. *The graded algebra* $\mathrm{Gr}(B)$ *associated to* $\mathcal{F} = \{\mathcal{F}_\alpha = \pi(\Im_\alpha)\}_{\alpha \in \mathbb{Z}}$ *is isomorphic to*

$$\mathrm{Gr}(B) \simeq \mathbf{k}[X, Y, Z, S, T]/\widehat{\mathfrak{b}} = \mathbf{k}[X, Y, Z, S, T]/\langle X^n Y - S^d - T^r,\, Y^m - X^e Z \rangle.$$

Furthermore, denote by $B_{[i]} = \mathcal{F}_i / \mathcal{F}_{i-1}$. *Then:*
(1) $B_{[-i]} = \mathbf{k}[\widehat{s}, \widehat{t}]\widehat{x}^i$ *for* $i > 0$,
(2) $B_{[0]} = \mathbf{k}[\widehat{s}, \widehat{t}]$,
(3) $B_{[nj-l]} = \mathbf{k}[\widehat{s}, \widehat{t}]\widehat{x}^l \widehat{y}^j$ *for* $l \in \{0, \ldots, n-1\}$ *and* $j \in \{0, \ldots, m-1\}$,
(4) $B_{[(nm+e)i-l]} = \mathbf{k}[\widehat{s}, \widehat{t}]\widehat{x}^l \widehat{z}^i$ *for* $l \in \{0, \ldots, e-1\}$ *and* $i \in \mathbb{N}$,
(5) $B_{[(nm+e)i+nj-l]} = \mathbf{k}[\widehat{s}, \widehat{t}]\widehat{x}^l \widehat{y}^j \widehat{z}^i$, *for* $l \in \{0, \ldots, \min\{n, e\} - 1\}$ *and* $i, j \in \mathbb{N}\backslash\{0\}$.

2.3. **The Derksen invariant and degree of derivations.**
Recall that the *Derksen invariant* of a $\mathbf{k}$-domain A is defined to be the sub-algebra $\mathcal{D}(A) \subset A$ generated by the kernels of all non-zero locally nilpotent derivation of A. That is, $\mathcal{D}(A) := \mathbf{k}[\cup_{\partial \in \mathrm{LND}(A)\backslash\{0\}} \ker \partial] \subset A$. The following theorem determines the Derksen invariant for the class 2.

Theorem 2.6. *Let B be the $\mathbf{k}$-domain defined by $B := \mathbf{k}[x, y, z, t] \simeq \mathbf{k}[X, Y, Z, T]/\langle X^n Y - (Y^m - X^e Z)^d - T^r - X\,Q(X, Y^m - X^e Z, T)\rangle$. Then $\mathcal{D}(B) = \mathbf{k}[x, s, t]$ where $s = y^m - x^e z$. In particular, B is not algebraically isomorphic to $\mathbb{A}^3_{\mathbf{k}}$.*

Proof. Given a non-zero $\partial \in \mathrm{LND}(B)$, by Proposition 1.8 (3) and (4), it respects the ω_B-filtration determined in Proposition 2.4. Therefore, it induces a non-zero locally nilpotent derivation $\widehat{\partial} := \mathrm{gr}_{\omega_B}(\partial)$ of $\mathrm{Gr}(B)$. Suppose that $f \in \ker \partial$, then $\widehat{f} := \mathrm{gr}(f) \in \ker \widehat{\partial}$ is an homogenous element of $\mathrm{Gr}(B)$.

Assume that $\widehat{f} \notin \mathbf{k}[\widehat{x}, \widehat{s}, \widehat{t}]$, then, by Corollary 2.5, $\widehat{y}$ or $\widehat{z}$ must divides $\widehat{f}$. This yields a contradiction as follows.

If $\widehat{z}$ divides $\widehat{f}$, then $\widehat{\partial}(\widehat{z}) = 0$ as $\ker \widehat{\partial}$ is factorially closed. On the other hand, if $\widehat{y}$ divides $\widehat{f}$, then $\widehat{y} \in \ker \widehat{\partial}$. Thus, the relation $\widehat{y}^m - \widehat{x}^e \widehat{z}$ implies that $\widehat{z} \in \ker \widehat{\partial}$ as $\ker \widehat{\partial}$ is factorially closed. Therefore, either way the assumption $\widehat{f} \notin \mathbf{k}[\widehat{x}, \widehat{s}, \widehat{t}]$ implies that $\widehat{\partial}(\widehat{z}) = 0$.

The case where $e = 1$ is particular since then $\widehat{\partial}$ extends to a locally nilpotent derivation of the $\mathbf{k}(\widehat{z})$-domain $\mathbf{k}(\widehat{z})[X, Y, S, T]/\langle X^n Y - S^d - T^r,\, Y^m - X\widehat{z} \rangle$, which is isomorphic to

$$\mathbf{k}(\widehat{z})[Y, S, T]/\langle \frac{1}{\widehat{z}^n} Y^{nm+1} - S^d - T^r \rangle.$$

Since the latter is a rigid ring, see [3, Section 7.1], we get $\widehat{\partial} = 0$, a contradiction.

For the case where $e>1$, let $\varpi \in \mathbb{Z}^5$ be another weight degree function on $\mathrm{Gr}(B)$ defined by:

$$\varpi(X)=q,\ \varpi(Y)=-n_0,\ \varpi(Z)=-mn_0-eq,\ \varpi(S)=r,\ \varpi(T)=d,$$

where $rd=nq-n_0$, $q\in\mathbb{Z}$, and $n_0\in\{0,\ldots,n-1\}$. Then, $\mathrm{Gr}_{\varpi}(\mathrm{Gr}(B))=\mathrm{Gr}(B)$, that is, ϖ is a $\mathbb{Z}$-grading of $\mathrm{Gr}(B)$. Hence, $\widehat{\partial}$ induces $\widetilde{\partial}:=\mathrm{gr}_{\varpi}(\widehat{\partial})$ a non-zero locally nilpotent derivation of $\mathrm{Gr}(B)$, such that $\widetilde{\partial}(\widehat{z})=0$. Choose $H\in\ker\widetilde{\partial}$ which is a non-constant homogeneous, relative to both grading of B, and algebraically independent of $\widehat{z}$, which is possible since $\ker\widetilde{\partial}$ is generated by homogeneous elements and it is of transcendence degree 2 over $\mathbf{k}$, see [18]. Then, the only possibility for H is $H=h(\widehat{s},\widehat{t})$ otherwise we get $\widetilde{\partial}=0$.
Since $\gcd(d,r)=1$, there exists a standard homogeneous polynomial $h_0\in\mathbf{k}^{[2]}$ such that $h(\widehat{s},\widehat{t})=h_0(\widehat{s}^d,\widehat{t}^r)$, see [10, Lemma 4.6]. Thus we have $h_0(\widehat{s}^d,\widehat{t}^r)\in\ker\widetilde{\partial}$, which implies that either $\widetilde{\partial}(\widehat{s})=0$ or $\widetilde{\partial}(\widehat{t})=0$ (or both), see [10, Prop. 9.4]. But if $\widetilde{\partial}(\widehat{t})=0$, then $\widetilde{\partial}$ extends to a locally nilpotent derivation of

$$A:=\mathbf{k}(\widehat{z},\widehat{t})[X,Y,S]/\langle X^nY-S^d-\widehat{t}^r,\ Y^m-X^e\widehat{z}\rangle.$$

It follows from the Jacobian criterion that $\mathrm{Spec}(A)$ has a non-empty set of singular points as $e>1$. Since A is an integral domain of transcendence degree one over $\mathbf{k}(\widehat{z},\widehat{t})$, [10, Corollary 1.29] implies that A is rigid, and therefore $\widetilde{\partial}=0$, a contradiction. In the same way we get a contradiction if $\widetilde{\partial}(\widehat{s})=0$.

Therefore, the only possibility is that $\widehat{f}\in\mathbf{k}[\widehat{x},\widehat{s},\widehat{t}]$, which yields $f\in\mathbf{k}[x,s,t]$. This proves that $\mathcal{D}(B)\subseteq\mathbf{k}[x,s,t]\simeq\mathbf{k}^{[3]}$. To complete the proof, define $D_1,D_2\in\mathrm{LND}(B)$ by:

$$D_1(x)=D_1(t)=0, D_1(s)=x^{n+e}, D_1(y)=x^e(ds^{d-1}+x\frac{\partial Q}{\partial s}),\ D_1(z)=my^{m-1}(ds^{d-1}+x\frac{\partial Q}{\partial s})-x^n$$

and

$$D_2(x)=D_2(s)=0, D_2(t)=x^{n+e}, D_2(y)=x^e(rt^{r-1}+x\frac{\partial Q}{\partial t}),\ D_2(z)=my^{m-1}(rt^{r-1}+x\frac{\partial Q}{\partial t}).$$

Then obviously $\mathbf{k}[x,s,t]\subseteq\mathcal{D}(B)$. □

What we did establish in the proof of Theorem 2.6 is actually more than the assertion announced in the Theorem itself. Indeed,

Lemma 2.7. *Let ω_B be the degree function on B defined as in Theorem 2.6, then:*

$$\deg_{\omega_B}\partial\leq -n-e\,;\ \textit{for every non-zero}\ \partial\in\mathrm{LND}(B).$$

Proof. Let $\partial\in\mathrm{LND}(B)$ be non-zero. Continuing the notation of the proof of Theorem 2.6, ∂ induces $\widehat{\partial}:=\mathrm{gr}(\partial)$ a non-zero locally nilpotent derivation of $\mathrm{Gr}(B)$. We have established that $\widehat{\partial}(\widehat{z})\neq 0$. Denote by τ the degree of ∂ with respect to ω_B, $\tau:=\deg_{\omega_B}\partial$.

Assume for contradiction that $\tau=\deg_{\omega_B}\partial>-(n+e)$. Then, for every $b\in B$ such that $\omega_B(b)=i$, we have by definition of $\widehat{\partial}$ that $\widehat{\partial}(\hat{b})=\begin{cases}0 & ;\ \text{if } \omega_B(\partial(b))<i+\tau\\ \widehat{\partial(b)} & ;\ \text{if } \omega_B(\partial(b))=i+\tau\end{cases}$. Thus we conclude that either $\widehat{\partial}(\widehat{z})=0$, which is excluded, or $\widehat{\partial}(\widehat{z})=\widehat{\partial(z)}$. But since $\omega_B(z)=nm+e$, we see that $\partial(z)\in\mathcal{F}_{nm+e+\tau}$, and $\widehat{\partial(z)}\in B_{[nm+e+\tau]}$. So $\widehat{z}$ divides $\widehat{\partial}(\widehat{z})$ by Corollary 2.5, which implies that $\widehat{\partial}(\widehat{z})=0$ by reasons of degree, see [10, Corollary 1.20], which is absurd. Therefore, the only possibility is that $\tau=\deg_{\omega_B}\partial\leq -n-e$. And we are done. □

Consider the following chain of inclusions:

$$\mathcal{D}(B)=\mathbf{k}[x,s,t]\hookrightarrow R=\mathbf{k}[x,s,t,y]\hookrightarrow B=\mathbf{k}[x,s,t,y,z],$$

where R is the Russell $\mathbf{k}$-domain corresponding to the pair $(n,S^d+T^r+X\,Q(X,S,T))$, we have the following.

Corollary 2.8. *Every $\partial\in\mathrm{LND}(B)$ restricts to a locally nilpotent derivation of $\mathbf{k}[x,y,s,t]=R$ (resp. $\mathbf{k}[x,s,t]=\mathcal{D}(B)$). Furthermore,*

$$\partial(R)\subseteq\langle x^e\rangle_R\quad\textit{and}\quad\partial(\mathcal{D}(B))\subseteq\langle x^{n+e}\rangle_{\mathcal{D}(B)}\,,$$

where $\langle x^e\rangle_R$ (resp. $\langle x^{n+e}\rangle_{\mathcal{D}(B)}$) is the principle ideal of R (resp.$\mathcal{D}(B)$) generated by x^e (resp. x^{n+e}).

Proof. Let $\partial \in \mathrm{LND}(B)$ be non-zero. By Lemma 2.7, we have $\tau = \deg_{\omega_B} \partial \leq -n-e$. This means $\partial(\mathcal{F}_i) \subseteq \mathcal{F}_{i+\tau} \subseteq \mathcal{F}_i$, and hence, $\partial(\mathcal{D}(B)) \subseteq \mathcal{D}(B)$ and $\partial(R) \subseteq R$. Furthermore,

$$\partial(\mathbf{k}[x,s,t]) \subseteq \mathcal{F}_{-n-e} = \mathbf{k}[x,s,t]x^{n+e} + \mathcal{F}_{-n-e-1} = \langle x^{n+e} \rangle_{\mathbf{k}[x,s,t]}.$$

Finally,

$$\partial(y) \in \mathbf{k}[x,s,t]x^e + \mathcal{F}_{-e-1} = \langle x^e \rangle_{\mathbf{k}[x,s,t]}.$$

The latter implies that $\partial(R) \subseteq \langle x^e \rangle_R$, as desired. □

2.4. **The Makar-Limanov invariant and** LND.

Recall that the *Makar-Limanov invariant* $\mathrm{ML}(A)$ of a $\mathbf{k}$-domain A is defined to be the intersection of the kernels of all locally nilpotent derivations of A. That is, $\mathrm{ML}(A) := \cap_{\partial \in \mathrm{LND}(A)} \ker \partial$.

The observation that every locally nilpotent derivation of B must restrict to a locally nilpotent derivation of the sub-algebra R, introduce a consecutive way to compute the Makar-Limanov invariant. That is, consider the inclusion $R \hookrightarrow B$. It is well-known that $\mathrm{ML}(R) = \mathbf{k}[x]$; $n \geq 2$, see [12, 11, 17]. On the other hand, by Theorem 2.8, every $\partial \in \mathrm{LND}(B)$ restricts to $\partial|_R$ a locally nilpotent derivation of R. Therefore, since $\mathrm{ML}(R) = \cap_{D \in \mathrm{LND}(R)} \ker D \subseteq \cap_{\partial \in \mathrm{LND}(B)} \ker \partial|_R$, we immediately obtain $\mathrm{ML}(R) = \mathbf{k}[x] \subseteq \mathrm{ML}(B)$. Finally, since $\mathrm{ML}(B) \subseteq \ker D_1 \cap \ker D_2 = \mathbf{k}[x]$ where $D_1, D_2 \in \mathrm{LND}(B)$ define as in the proof of Theorem 2.6, we get Corollary 2.9, in the cases $n \geq 2$, for free.

Nevertheless, for the general case, we present an alternative approach to compute the Makar-Limanov invariant for the class of examples 2. That is, it can be deduced from Corollary 2.8 as follows.

Corollary 2.9. $\mathrm{ML}(B) = \mathbf{k}[x]$.

Proof. Let $\partial \in \mathrm{LND}(B)$, then Theorem 2.8 in particular, asserts that $\partial(\mathbf{k}[x,s,t]) \subseteq \langle x^{n+e} \rangle_{\mathbf{k}[x,s,t]}$. This implies that $\partial(x)$ is divisible by x, thus by reasons of degree, see also [10, Corollary 1.20], we conclude that $\partial(x) = 0$ and $\mathbf{k}[x] \subseteq \mathrm{ML}(B)$. Finally, noting that $\mathrm{ML}(B) \subseteq \mathbf{k}[x] = \ker D_1 \cap \ker D_2$ where $D_1, D_2 \in \mathrm{LND}(B)$ define as in the proof of Theorem 2.6, we have $\mathrm{ML}(B) = \mathbf{k}[x]$, as desired. □

The following Corollary, which is a consequence of Corollary 2.8, describes the set $\mathrm{LND}(B)$. Denote by $\mathrm{LND}_{\mathbf{k}[x]}(\mathbf{k}[x,s,t])$ the set of locally nilpotent derivations of $\mathbf{k}[x,s,t] \simeq \mathbf{k}^{[3]}$ that have x in their kernels, then:

Corollary 2.10. $\mathrm{LND}(B) = x^e\left(\mathrm{LND}(R)\right) = x^{n+e}\left(\mathrm{LND}_{\mathbf{k}[x]}(\mathbf{k}[x,s,t])\right)$.

Proof. Let δ be a locally nilpotent derivation of R (resp. $\mathbf{k}[x,s,t]$ that annihilates x), then the derivation $x^e\delta$ (resp. $x^{n+e}\delta$) extends to a locally nilpotent derivation of B by taking

$$(x^e\delta)(z) = \frac{(x^e\delta)(y^m) - (x^e\delta)(s)}{x^e} = \delta(y^m) - \delta(s)$$

resp.

$$(x^{n+e}\delta)(y) = \frac{(x^{n+e}\delta)(s^d + t^r + xQ)}{x^n} = x^e\delta(s^d + t^r + xQ)\ , \text{ and}$$

$$(x^{n+e}\delta)(z) = \frac{(x^{n+e}\delta)(y^m - s)}{x^e} = my^{m-1}\delta(s^d + t^r + xQ) - x^n\delta(s).$$

We denote $\widetilde{\delta} = x^e\delta$ (resp. $\widetilde{\delta} = x^{n+e}\delta$). Conversely, Corollary 2.8 ensures that every $\partial \in \mathrm{LND}(B)$ restricts to $\partial|_R \in \mathrm{LND}(B)$ as well as $\partial|_{\mathbf{k}[x,s,t]} \in \mathrm{LND}_{\mathbf{k}[x]}(\mathbf{k}[x,s,t])]$, such that $\partial|_R = x^e\delta_1$ and $\partial|_{\mathbf{k}[x,s,t]} = x^{n+e}\delta_2$ for some $\delta_1 \in \mathrm{LND}(R)$ and $\delta_2 \in \mathrm{LND}_{\mathbf{k}[x]}(\mathbf{k}[x,s,t])$, hence $\delta_1|_{\mathbf{k}[x,s,t]} = x^n\delta_2$. This establishes the correspondence. Finally, it is straightforward to check that the latter is a one-to-one correspondence, that is, $\widetilde{\partial|}_R = \partial$ and $\widetilde{\delta}|_R = \delta$ (resp. $\widetilde{\partial|}_{\mathbf{k}[x,s,t]} = \partial$ and $\widetilde{\delta}|_{\mathbf{k}[x,s,t]} = \delta$). And we are done. □

The next Corollary describes the kernels of locally nilpotent derivations of B. The proof of [10, Corollary 9.8] also applies here.

Corollary 2.11. *Let $\partial \in \mathrm{LND}(B)$ be non-zero, then there exists $F \in \mathbf{k}[x,s,t] \subset B$ such that F is a $\mathbf{k}(x,x^{-1})$-variable of $\mathbf{k}(x,x^{-1})[s,t]$, and* $\ker \partial = \mathbf{k}[x,F] = \mathbf{k}^2$.

3. Exponential Modifications

3.1. Definitions and basic properties.

Let A be a finitely generated domain over a field $\mathbf{k}$ of characteristic zero, I be an ideal in A and f be a non-zero element of I.

Definition 3.1. By the *affine modification* of A along f with center I, see [15, 13, 21], we mean the sub-algebra $A' := A[I/f]$ of A_f (the localization of A with respect to f) generated by A and the sub-set I/f. Similarly, for any $\mathbf{k}$-domain A, the sub-algebra $A' := A[I/f]$ of A_f is called the *modification* of A along f with center I. The pair (f, I) is called the *locus* of the modification and A is called the *base* of the modification.

If the ideal I is finitely generated, say $I = \langle f, b_1, \ldots, b_r\rangle_A$, then A' is the sub-algebra of $A_f \subset \mathrm{Frac}(A)$ which is generated by A and the elements $b_1/f, \ldots, b_r/f$. That is,

$$A[I/f] = \{P(b_1/f, \ldots, b_r/f);\ P(X_1, \ldots, X_r) \in A[X_1, \ldots, X_r]\}.$$

Therefore, we get

$$A[I/f] = \{a/f^d \in A_f;\ a \in I^d \text{ and } d \in \mathbb{N}\}.$$

The extension of the ideal I in $A' = A[I/f]$ coincides with the principal ideal generated by f, that is, $I.A[I/f] = \langle f\rangle_{A[I/f]}$.

The next lemma manifests the universal property of modifications, see [13, Proposition 2.1 and Corollary 2.2].

Lemma 3.2. *Let $\Psi : A \longrightarrow B$ be an isomorphism between domains A and B, I be an ideal in A, and $f \in I$. Then Ψ extends in a unique way to an isomorphism $\widetilde{\Psi} : A[I/f] \longrightarrow B[\Psi(I)/\Psi(f)]$.*

Proof. Define $\widetilde{\Psi} : A[I/f] \longrightarrow B[\Psi(I)/\Psi(f)]$ by $\widetilde{\Psi}(a) = \Psi(a)$ for every $a \in A$ and $\widetilde{\Psi}(P(b_1/f, \ldots, b_s/f)) = P_\Psi(\Psi(b_1)/\Psi(f), \ldots, \Psi(b_s)/\Psi(f))$, where $P(X_1, \ldots, X_s) = \sum_{\text{finite}} a_i X_1^{n_{(1,i)}} \ldots X_s^{n_{(s,i)}} \in A[X_1, \ldots, X_s]$ and $P_\Psi(X_1, \ldots, X_s) = \sum_{\text{finite}} \Psi(a_i) X_1^{n_{(1,i)}} \ldots X_s^{n_{(s,i)}} \in B[X_1, \ldots, X_s]$. Then $\widetilde{\Psi}$ is an isomorphism with inverse $\widetilde{\Psi}^{-1} : B[\Psi(I)/\Psi(f)] \longrightarrow A[I/f]$ defined by $\widetilde{\Psi}^{-1}(\Psi(a)) = \Psi^{-1}(\Psi(a)) = a$ for every $\Psi(a) \in B$ (i.e., $\widetilde{\Psi}^{-1}|_B = \Psi^{-1}$) and $\widetilde{\Psi}^{-1}(H(\Psi(b_1)/\Psi(f), \ldots, \Psi(b_s)/\Psi(f)) = H_{\Psi^{-1}}(b_1/f, \ldots, b_s/f)$. Finally, let Φ be an isomorphism between $A[I/f]$ and $B[\Psi(I)/\Psi(f)]$, such that $\Phi|_A = \Psi$, then $\Phi(P(b_1/f, \ldots, b_s/f)) = P_{\Phi|_A}(\Phi(b_1/f), \ldots, \Phi(b_s/f))$. Since $\Phi(b_i) = \Phi(f b_i/f) = \Phi(f)\Phi(b_i/f)$, we conclude that $\Phi(b_i/f) = \Phi(b_i)/\Phi(f)$ for every i. Hence $\Phi(P(b_1/f, \ldots, b_s/f)) = P_{\Phi|_A}(\Phi(b_1)/\Phi(f), \ldots, \Phi(b_s)/\Phi(f)) = P_\Psi(\Psi(b_1)/\Psi(f), \ldots, \Psi(b_s)/\Psi(f))$ and hence $\Phi = \widetilde{\Psi}$, as desired. $\square$

3.2. Exponential modifications .

We are interested in modifications of A along elements of the form $f = a^n$; $n \in \mathbb{N}\backslash\{0\}$ for some element a in A.

Definition 3.3. Let A be an integral domain, I be an ideal in A, and a be an irreducible element in A such that $a^n \in I$. The modification $A[I/a^n]$ of A along a^n with center I will be called the *exponential modification* of A with respect to a. The *contraction* of $\langle a^N\rangle_{A[I/a^n]}$ with respect to the inclusion $A \overset{\iota}{\hookrightarrow} A[I/a^n]$ (also called the contraction of $\langle a^N\rangle_{A[I/a^n]}$ in A) is $\langle a^N\rangle^{\mathbf{c}}_{A[I/a^n]} := \{b \in A;\ \iota(b) \in \langle a^N\rangle_{A[I/a^n]}\}$.

The principal ideal $\langle a^N\rangle_{A[I/a^n]}$ will be denoted simply by $\langle a^N\rangle$ (not to be confused with $\langle a^N\rangle_A$ the principle ideal in A generated by a^N, i.e., $\langle a^N\rangle_{A[I/a^n]} \neq \langle a^N\rangle_A$ in general). Note that the contraction of $\langle a^N\rangle$ in A coincides with

$$\langle a^N\rangle^{\mathbf{c}} = A \cap \langle a^N\rangle.$$

Also, the extension of I to $A[I/a^n]$ (i.e., the ideal in $A[I/a^n]$ generated by I) coincides with the principle ideal generated by a^n, that is, $I.A[I/a^n] = \langle a^N\rangle$.

Consider the following chain of principal ideals in $A[I/a^n]$:

$$A[I/a^n] = \langle 1\rangle \supset \langle a\rangle \supset \langle a^2\rangle \supset \cdots \supset \langle a^n\rangle,$$

it induces the following chain of ideals in A.

$$A = \langle 1\rangle^{\mathbf{c}} \supset \langle a\rangle^{\mathbf{c}} \supset \langle a^2\rangle^{\mathbf{c}} \supset \cdots \supset \langle a^n\rangle^{\mathbf{c}}.$$

Note that $\langle a^n\rangle^{\mathbf{c}} = I$. To the latter chain of ideals we associate the following chain of sub-algebras of $A[I/a^n] \subset A[a^{-1}]$:

$$A \subset A[\langle a\rangle^{\mathbf{c}}/a] \subset A[\langle a^2\rangle^{\mathbf{c}}/a^2] \subset \cdots \subset A[\langle a^n\rangle^{\mathbf{c}}/a^n] = A[I/a^n].$$

Note that $A[\langle a^N\rangle^{\mathbf{c}}/a^N]$ is the exponential modification of A, along a^N with center $\langle a^N\rangle^{\mathbf{c}}$ for every $N \in \mathbb{N}$.

Definition 3.4. The chain $A = \langle 1\rangle^{\mathbf{c}} \supset \langle a\rangle^{\mathbf{c}} \supset \langle a^2\rangle^{\mathbf{c}} \supset \cdots \supset \langle a^n\rangle^{\mathbf{c}} = I$ of ideals in A will be called the *contraction chain* associated to $A[I/a^n]$ and the chain $A \subset A[\langle a\rangle^{\mathbf{c}}/a] \subset A[\langle a^2\rangle^{\mathbf{c}}/a^2] \subset \cdots \subset A[\langle a^n\rangle^{\mathbf{c}}/a^n] = A[I/a^n]$ of sub-algebras of $A[I/a^n]$ will be called the *exponential chain* of $A[I/a^n]$.

The next theorem shows in particular that isomorphisms between exponential modifications, which preserve bases of modifications together with their principal ideals generated by their centers, respect the associated contraction and exponential chains.

Theorem 3.5. *Let $\Psi : A[I/a^n] \longrightarrow B'$ be an isomorphism between the exponential modifications $A[I/a^n]$ and B'. Assume that $\Psi(A) = B \subset B'$ and $\Psi(a) = b$. Then*:

(1) *Ψ respects the contraction chains, that is, $\Psi(\langle a^N\rangle^{\mathbf{c}}_{A[I/a^n]}) = \langle b^N\rangle^{\mathbf{c}}_{B'}$ for every $N \in \mathbb{N}$.*

(2) *Ψ respects the exponential chains, that is, $\Psi(A[\langle a^N\rangle^{\mathbf{c}}_{A[I/a^n]}/a^N]) = B[\langle b^N\rangle^{\mathbf{c}}_{B'}/b^N]$ for every $N \in \mathbb{N}$.*

In particular, B' can be realized as the exponential modification of B with locus $(b^n, \langle b^n\rangle^{\mathbf{c}}_{B'})$, that is, $B' = B[\langle b^n\rangle^{\mathbf{c}}_{B'}/b^n]$.

Proof. Assertion (1), since $\Psi(a) = b$, $\Psi(A) = B$ and $\langle a^N\rangle^{\mathbf{c}}_{A[I/a^n]} = A \cap \langle a^N\rangle_{A[I/a^n]}$, we have $\Psi(\langle a^N\rangle^{\mathbf{c}}_{A[I/a^n]}) = B \cap \langle b^N\rangle_{B'}$ for every $N \in \mathbb{N}$. Since $B \cap \langle b^N\rangle_{B'} = \langle b^N\rangle^{\mathbf{c}}_{B'}$, we conclude that $\Psi(\langle a^N\rangle^{\mathbf{c}}_{A[I/a^n]}) = \langle b^N\rangle^{\mathbf{c}}_{B'}$ for every $N \in \mathbb{N}$.

Assertion (2), since $\Psi(A) = B$ and $\Psi(a) = b$, Lemma 3.2 asserts that the restriction $\Psi|_A$ of Ψ to A extends to an isomorphism $\Phi_N := \widetilde{\Psi|_A}$ between $A[\langle a^N\rangle^{\mathbf{c}}_{A[I/a^n]}/a^N]$ and $B[\Psi(\langle a^N\rangle^{\mathbf{c}}_{A[I/a^n]})/\Psi(a)^N]$ for every $N \in \mathbb{N}$. By assertion (1), the latter ring coincides with $B[\langle b^N\rangle^{\mathbf{c}}_{B'}/b^N]$. Moreover, noting that this extension is unique, Φ_N coincides with the restriction $\Psi|_{A[\langle a^N\rangle^{\mathbf{c}}_{A[I/a^n]}/a^N]}$ of Ψ to $A[\langle a^N\rangle^{\mathbf{c}}_{A[I/a^n]}/a^N]$, i.e., $\Phi_N = \Psi|_{A[\langle a^N\rangle^{\mathbf{c}}_{A[I/a^n]}/a^N]}$. Hence, $\Psi(A[\langle a^N\rangle^{\mathbf{c}}_{A[I/a^n]}/a^N]) = B[\langle b^N\rangle^{\mathbf{c}}_{B'}/b^N]$, for every $N \in \mathbb{N}$, and in particular $B' = B[\langle b^n\rangle^{\mathbf{c}}_{B'}/b^n]$, as desired. $\square$

The previous theorem asserts that if $A[I/a^n] \simeq B'$, then there exist an element $b \in B'$, a sub-algebra $B \subset B'$, and an ideal $J \subset B$ contains b^n, such that $A \simeq B$ and B' can be realized as the modification of B with locus $(b^n, J := \langle b^n\rangle^{\mathbf{c}}_{B'})$. Furthermore, every $\mathbf{k}$-isomorphism $\Psi : A[I/a^n] \longrightarrow B'$ such that $\Psi(A) = B$, restricts to a $\mathbf{k}$-isomorphism between $A[\langle a^N\rangle^{\mathbf{c}}_{A[I/a^n]}/a^N]$ and $B[\langle b^N\rangle^{\mathbf{c}}_{B'}/b^N]$ for every N, where $\Psi(a) = b$ and $\Psi(I) = J$. Therefore, we have the following commutative diagram:

$$\begin{array}{ccc}
A[I/a^n] & \xrightarrow{\Psi} & B' = B[\langle b^n\rangle^{\mathbf{c}}_{B'}/b^n] \\
\cup & \circlearrowleft & \cup \\
\vdots & \vdots & \vdots \\
\cup & \circlearrowleft & \cup \\
A[\langle a^2\rangle^{\mathbf{c}}_{A[I/a^n]}/a^2] & \xrightarrow{\sim} & B[\langle b^2\rangle^{\mathbf{c}}_{B'}/b^2] \\
\cup & \circlearrowleft & \cup \\
A[\langle a\rangle^{\mathbf{c}}_{A[I/a^n]}/a] & \xrightarrow{\sim} & B[\langle b\rangle^{\mathbf{c}}_{B'}/b] \\
\cup & \circlearrowleft & \cup \\
A & \xrightarrow[\Psi|_A]{\sim} & B
\end{array}$$

4. Isomorphism classes and Automorphism groups

Let $m, d, r \geq 2$ be fixed such that $\gcd(d, r) = 1$. For every $e \geq 0$, $n \geq 1$ such that $(n, e) \neq (1, 0)$, and every $Q \in \mathbf{k}[X, S, T]$, we denote by $B_{(n,e,Q)}$ the following $\mathbf{k}$-domain:

$$B_{(n,e,Q)} := \mathbf{k}[x, y, z, t] \simeq \mathbf{k}[X, Y, Z, T]/\langle X^nY - (Y^m - X^eZ)^d - T^r - X\,Q(X, Y^m - X^eZ, T)\rangle$$

which is isomorphic to

$$\mathbf{k}[X, Y, Z, S, T]/\langle X^nY - S^d - T^r - X\,Q(X, S, T),\ Y^m - X^eZ - S\rangle.$$

Also, we denote by $\mathbf{R}_{(n,S^d+T^r+XQ)}$ the Russel $\mathbf{k}$-domain:

$$\mathbf{R}_{(n,S^d+T^r+XQ)} := \mathbf{k}[x,y,s,t] \simeq \mathbf{k}[X,Y,S,T]/\langle X^nY - S^d - T^r - X\,Q(X,S,T)\rangle$$

Consider the following two chains of inclusions, for $i \in \{1,2\}$:

$$\mathbf{k}[x,s,t] \hookrightarrow \mathbf{R}_{(n_i,S^d+T^r+XQ_i)} = \mathbf{k}[x,s,t,y_i] \hookrightarrow B_{(n_i,e_i,Q_i)} = \mathbf{k}[x,s,t,y_i,z_i] \hookrightarrow B_{(n_i,e_i,Q_i)}[x^{-1}] = \mathbf{k}[x,x^{-1},s,t].$$

The last inclusion is realized by the localization homomorphism with respect to x, where

$$y_i = x^{-n_i}(s^d+t^r+xQ_i),\ z_i = x^{-n_im-e_i}((s^d+t^r+xQ_i)^m - x^{n_im}s) \in \mathbf{k}[x,x^{-1},s,t]\ ;\ \ i \in \{1,2\}.$$

Theorem 2.6 and Corollary 2.9 implies that $\mathcal{D}(B_{(n_i,e_i,Q_i)}) = \mathcal{D}(\mathbf{R}_{(n_i,S^d+T^r+XQ_i)}) = \mathbf{k}[x,s,t] \simeq \mathbf{k}^{[3]}$ and $\mathrm{ML}(B_{(n_i,e_i,Q_i)}) = \mathrm{ML}(\mathbf{R}_{(n_i,S^d+T^r+XQ_i)}) = \mathbf{k}[x]$.

4.1. Basic facts.

Some conditions that two $\mathbf{k}$-domains, with the same Derksen and Makar-Limanov invariants, must verify to be isomorphic can be deduced from properties of their locally nilpotent derivations. Indeed, the next proposition shows how a prior knowledge of degrees of all locally nilpotent derivations relative to some degree function can be used to obtain some conditions that two $\mathbf{k}$-domains must satisfy to be isomorphic.

Proposition 4.1. *Let $\Psi : B_{(n_1,e_1,Q_1)} \longrightarrow B_{(n_2,e_2,Q_2)}$ be a $\mathbf{k}$-isomorphism. Then:*

1. *There exists $\lambda \in \mathbf{k}\setminus\{0\}$ such that $\Psi(x) = \lambda x$.*
2. *$n_1 + e_1 = n_2 + e_2$.*

Proof. Since every $\mathbf{k}$-isomorphism Ψ between $B_{(n_1,e_1,Q_1)}$ and $B_{(n_2,e_2,Q_2)}$ must preserve the Makar-Limanov and the Derksen invariants, we deduce by virtue of Corollary 2.9 and Theorem 2.6 that Ψ restricts to a $\mathbf{k}$-automorphism of $\mathbf{k}[x]$ (resp. $\mathbf{k}[x,s,t] \simeq \mathbf{k}^{[3]}$). This implies that $\Psi(x) = \lambda x + c$ for some $\lambda \in \mathbf{k}\setminus\{0\}$ and $c \in \mathbf{k}$, and that $\Psi(s), \Psi(t) \in \mathbf{k}[x,s,t]$.
Let $\partial_1 \in \mathrm{LND}(B_{(n_1,e_1,Q_1)})$ be a non-zero, then $\partial_2 := \Psi\partial_1\Psi^{-1}$ is also a non-zero locally nilpotent derivation of $B_{(n_2,e_2,Q_2)}$. On the other hand, Corollary 2.8 ensures that ∂_i restricts to $\mathbf{k}[x,s,t]$ in such a way that $\partial_i(\mathbf{k}[x,s,t]) \subseteq \langle x^{n_i+e_i}\rangle_{\mathbf{k}[x,s,t]} = x^{n_i+e_i}.\mathbf{k}[x,s,t]$ for every $i \in \{1,2\}$.
Define $\partial_1 \in \mathrm{LND}(B_{(n_1,e_1,Q_1)})$ by:

$$\partial_1(x) = \partial_1(t) = 0, \partial_1(s) = x^{n_1+e_1}, \partial_1(y_1) = x^{e_1}(ds^{d-1} + x\frac{\partial Q_1}{\partial s}),\ \partial_1(z_1) = my_1^{m-1}(ds^{d-1} + x\frac{\partial Q_1}{\partial s}) - x^{n_1}.$$

Then $\partial_2 := \Psi\partial_1\Psi^{-1} \in \mathrm{LND}(B_{(n_2,e_2,Q_2)})$ and we have $\partial_2\Psi = \Psi\partial_1$. Therefore, we obtain the relation $(\partial_2\Psi)(s) = (\Psi\partial_1)(s)$, where the second part is $\Psi\partial_1(s) = \Psi(x^{n_1+e_1}) = (\lambda x + c)^{n_1+e_1}$. As discussed before $\Psi(s) \in \mathbf{k}[x,s,t]$ and $\partial_2(\mathbf{k}[x,s,t]) \subseteq x^{n_2+e_2}.\mathbf{k}[x,s,t]$, thus the first part of the forgoing relation is $\partial_2(\Psi(s)) = x^{n_2+e_2}g(x,s,t)$ for some $g \in \mathbf{k}[x,s,t]$. Therefore, we get $x^{n_2+e_2}g(x,s,t) = (\lambda x + c)^{n_1+e_1}$ in $\mathbf{k}[x,s,t]$, which means that $x^{n_2+e_2}$ divides $(\lambda x + c)^{n_1+e_1}$ in $\mathbf{k}[x]$. This is possible if and only if $c = 0$, hence (1) follows, and $n_2 + e_2 \leq n_1 + e_1$. Finally, by symmetry we get $n_1 + e_1 = n_2 + e_2$, as desired. ☐

As a special case of Proposition 4.1, we have the following.

Corollary 4.2. *Let $\Psi : \mathbf{R}_{(n_1,S^d+T^r+XQ_1)} \longrightarrow \mathbf{R}_{(n_2,S^d+T^r+XQ_2)}$ be a $\mathbf{k}$-isomorphism. Then:*

1. *There exists $\lambda \in \mathbf{k}\setminus\{0\}$ such that $\Psi(x) = \lambda x$.*
2. *$n_1 = n_2$.*

Remark 4.3. Assertion (1) of Corollary 4.2 is well-known due to P. Russell. Nevertheless, assertion (2) is new. The proof of Proposition 4.1 present an alternative proof for assertion (1), using properties of locally nilpotent derivation, that delivers assertion (2) for free.

4.2. The chain of invariant sub-algebras associated to $B_{(n,e,Q)}$.

Let I be the ideal in $\mathbf{k}[x,s,t]$ generated by x^{nm+e}, $x^{n(m-1)+e}(s^d+t^r+xQ)$, and $(s^d+t^r+xQ)^m - x^{nm}s$, that is,

$$I = \left\langle x^{nm+e},\ x^{n(m-1)+e}(s^d+t^r+xQ),\ (s^d+t^r+xQ)^m - x^{nm}s\right\rangle_{\mathbf{k}[x,s,t]}.$$

Then, the affine modification of $\mathcal{D}(B_{(n,e,Q)}) = \mathbf{k}[x,s,t]$ along x^{nm+e} with center I is by definition

$$\mathbf{k}[x,s,t]\left[I/x^{nm+e}\right]$$

where $I/x^{nm+e} = x^{-nm-e}.I$ is the sub-set of $\mathbf{k}[x,x^{-1},s,t]$ consists of elements $x^{-nm-e}b$ where $b \in I$. Therefore,

$$\mathbf{k}[x,s,t]\left[I/x^{nm+e}\right] = \mathbf{k}[x,s,t][(s^d+t^r+xQ)/x^n, ((s^d+t^r+xQ)^m - x^{nm}s)/x^{nm+e}] = \mathbf{k}[x,s,t,y,z] = B.$$

That is,

Proposition 4.4. *The $\mathbf{k}$-domain $B_{(n,e,Q)}$ is the exponential modification of its Derksen invariant $\mathbf{k}[x,s,t]$ along x^{nm+e} with center I.*

The contraction chain associated to $B_{(n,e,Q)}$ is

$$\mathbf{k}[x,s,t] = \langle 1\rangle^{\mathbf{c}} \supset \langle x\rangle^{\mathbf{c}} \supset \langle x^2\rangle^{\mathbf{c}} \supset \cdots \supset \langle x^{nm+e}\rangle^{\mathbf{c}}.$$

The exponential chain of $B_{(n,e,Q)} \subset \mathbf{k}[x^{-1},x,s,t]$ is

$$\mathbf{k}[x,s,t] \subset \mathbf{k}[x,s,t][\langle x\rangle^{\mathbf{c}}/x] \subset \cdots \subset \mathbf{k}[x,s,t][\langle x^{nm+e}\rangle^{\mathbf{c}}/x^{nm+e}] = B_{(n,e,Q)} \subset \mathbf{k}[x^{-1},x,s,t].$$

Consider again the following two chains of inclusions, for $i \in \{1,2\}$:

$$\mathbf{k}[x,s,t] \hookrightarrow B_{(n_i,e_i,Q_i)} = \mathbf{k}[x,s,t,y_i,z_i] \hookrightarrow B_{(n_i,e_i,Q_i)}[x^{-1}] = \mathbf{k}[x,x^{-1},s,t].$$

Denote by

$$I_i = \left\langle x^{n_i m_i+e_i},\ x^{n_i(m_i-1)+e_i}(s^d+t^r+xQ_i),\ (s^d+t^r+xQ_i)^{m_i} - x^{n_i m_i}s \right\rangle_{\mathbf{k}[x,s,t]},$$

and $\langle x^N\rangle^{\mathbf{c}}_{B_{(n_1,e_1,Q_1)}}$ (resp. $\langle x^N\rangle^{\mathbf{c}}_{B_{(n_2,e_2,Q_2)}}$) the contraction of the ideal $\langle x^N\rangle_{B_{(n_1,e_1,Q)}}$ (resp. $\langle x^N\rangle_{B_{(n_2,e_2,Q)}}$) in $\mathbf{k}[x,s,t]$.

Via the previous description, we have the following.

Theorem 4.5. *Let $\Psi : B_{(n_1,e_1,Q_1)} \longrightarrow B_{(n_2,e_2,Q_2)}$ be a $\mathbf{k}$-isomorphism, then Ψ respects their contraction and exponential chains, that is,*

(1) $\Psi\left(\langle x^N\rangle^{\mathbf{c}}_{B_{(n_1,e_1,Q_1)}}\right) = \langle x^N\rangle^{\mathbf{c}}_{B_{(n_2,e_2,Q_2)}}$ *for every $N \in \mathbb{N}$. In particular, $\Psi(I_1) = I_2$.*
(2) $\Psi(\mathbf{k}[x,s,t][\langle x^N\rangle^{\mathbf{c}}_{B_{(n_1,e_1,Q_1)}}/x^N]) = \mathbf{k}[x,s,t][\langle x^N\rangle^{\mathbf{c}}_{B_{(n_2,e_2,Q_2)}}/x^N]$ *for every $N \in \mathbb{N}$. In particular, $B_{(n_2,e_2,Q_2)} = \mathbf{k}[x,s,t][\langle x^{n_1m_1+e_1}\rangle^{\mathbf{c}}_{B_{(n_2,e_2,Q_2)}}/x^{n_1m_1+e_1}]$.*

Proof. Theorem 2.6 and Proposition 4.1 imply that Ψ restricts to a $\mathbf{k}$-automorphism of $\mathbf{k}[x,s,t]$ and that $\Psi(x) = \lambda x$. Therefore, assertion (1) and (2) follow directly from Theorem 3.5. □

Therefore, we have the following commutative diagram:

$$\begin{array}{ccc}
B_{(n_1,e_1,Q_1)} = \mathbf{k}[x,s,t,y_1,z_1] & \xrightarrow{\Psi} & B_{(n_2,e_2,Q_2)} = \mathbf{k}[x,s,t,y_2,z_2] \\
\cup & \circlearrowright & \cup \\
\vdots & \vdots & \vdots \\
\cup & \circlearrowright & \cup \\
\mathbf{k}[x,s,t][\langle x^2\rangle^{\mathbf{c}}_{B_{(n_1,e_1,Q_1)}}/x^2] & \xrightarrow{\sim} & \mathbf{k}[x,s,t][\langle x^2\rangle^{\mathbf{c}}_{B_{(n_2,e_2,Q_2)}}/x^2] \\
\cup & \circlearrowright & \cup \\
\mathbf{k}[x,s,t][\langle x\rangle^{\mathbf{c}}_{B_{(n_1,e_1,Q_1)}}/x] & \xrightarrow{\sim} & \mathbf{k}[x,s,t][\langle x\rangle^{\mathbf{c}}_{B_{(n_2,e_2,Q_2)}}/x] \\
\cup & \circlearrowright & \cup \\
\mathbf{k}[x,s,t] & \xrightarrow{\sim} & \mathbf{k}[x,s,t]
\end{array}$$

In particular, the exponential chain $\mathbf{k}[x,s,t] \subset \mathbf{k}[x,s,t][\langle x\rangle^{\mathbf{c}}/x] \subset \cdots \subset \mathbf{k}[x,s,t][\langle x^{nm+e}\rangle^{\mathbf{c}}/x^{nm+e}] = B_{(n,e,Q)}$ characterizes $B_{(n,e,Q)}$. That is,

Corollary 4.6. *The exponential chain $\mathbf{k}[x,s,t] \subset \mathbf{k}[x,s,t][\langle x\rangle^{\mathbf{c}}/x] \subset \cdots \subset \mathbf{k}[x,s,t][\langle x^{nm+e}\rangle^{\mathbf{c}}/x^{nm+e}] = B_{(n,e,Q)}$ of $B_{(n,e,Q)}$ is invariant by every $\mathbf{k}$-automorphism Ψ of $B_{(n,e,Q)}$. That is, every $\Psi \in \mathrm{Aut}_{\mathbf{k}}(B_{(n,e,Q)})$ restricts to a $\mathbf{k}$-automorphism of every member of the exponential chain.*

4.3. **Computing contraction and exponential chains associated to $B_{(n,e,Q)}$.**
The next lemma describes the contraction of the ideal $\langle x^N\rangle$ in $\mathbf{k}[x,s,t]$ for every $N\in\mathbb{N}$.

Lemma 4.7. *Denote $F := s^d + t^r + xQ$, and $G := (s^d + t^r + xQ)^m - x^{nm}s = F^m - x^{nm}s$. Then,*

(1) $\langle x^{n_0}\rangle^{\mathbf{c}} = \langle F,\, x^{n_0}\rangle_{\mathbf{k}[x,s,t]}$ *for every* $n_0\in\{1,\dots,n\}$.
(2) $\langle x^{nm_0+e_0}\rangle^{\mathbf{c}} = \langle F^{m_0+1}, x^{n_0}F^{m_0},\dots,x^{(m_0-1)n+n_0}F, x^{nm_0+n_0}\rangle_{\mathbf{k}[x,s,t]}$ *for every* $m_0\in\{1,\dots,m-1\}$ *and* $n_0\in\{1,\dots,n\}$.
(3) $\langle x^{nm+e_0}\rangle^{\mathbf{c}} = \langle G,\, x^{e_0}F^m, x^{n+e_0}F^{m-1},\dots,x^{(m-1)n+e_0}F, x^{nm+e_0}\rangle_{\mathbf{k}[x,s,t]}$ *for every* $e_0\in\{1,\dots,e\}$.

Proof. We only prove (3) for the special case where $e_0 = e$, the rest can be proved in the same way. The proof is basically a consequence of the full description of the proper $\mathbb{Z}$-filtration defined on $B_{(n,e,Q)}$ as in Definition 2.2.
Let $\omega_{B_{(n,e,Q)}}$ be the degree function on $B_{(n,e,Q)}$ defined as in Definition 2.2, and suppose that $f\in\langle x^{nm+e}\rangle^{\mathbf{c}}$. Then $f\in\mathbf{k}[x,s,t]\cap\langle x^{nm+e}\rangle$ and there exists $b\in B_{(n,e,Q)}$ such that $f = x^{nm+e}b$. On the other hand, since $f\in\mathbf{k}[x,s,t]$, we have $\omega_{B_{(n,e,Q)}}(f)\le 0$. Noting that $\omega_{B_{(n,e,Q)}}(x^{nm+e}) = -nm-e$, we deduce that $\omega_{B_{(n,e,Q)}}(b)\le nm+e$. Therefore, by Proposition 2.4, b can be expressed as follows.

$$b = z\,(\sum_{i=0}^{e-1} x^i f_i(s,t)) + \sum_{j=1}^{m} y^j (\sum_{l=0}^{n-1} x^l\, g_{(j,l)}(s,t)) + h(x,s,t).$$

Hence,

$$x^{nm+e}b = x^{nm+e}\,z\,(\sum_{i=0}^{e-1} x^i f_i(s,t)) + x^e\sum_{j=1}^{m} x^{nm-nj}\,x^{nj}y^j(\sum_{l=0}^{n-1} x^l\, g_{(j,l)}(s,t)) + x^{nm+e}\,h(x,s,t).$$

Thus,

$$x^{nm+e}b = G\,(\sum_{i=0}^{e-1} x^i f_i(s,t)) + x^e\sum_{j=1}^{m} x^{n(m-j)}\,F^j(\sum_{l=0}^{n-1} x^l\, g_{(j,l)}(s,t)) + x^{nm+e}h(x,s,t).$$

Therefore, we conclude that

$$x^{nm+e}b\in\langle x^{nm+e}, x^iG, x^{(m-j)n+e+l}F^j;\ i\in\{0,\dots,e-1\},\, l\in\{0,\dots,n-1\},\text{ and } j\in\{1,\dots,m\}\rangle_{\mathbf{k}[x,s,t]}$$

Finally,

$$\langle x^{nm+e}\rangle^{\mathbf{c}} = \langle x^{nm+e},\, G,\, x^{(m-j)n+e}F^j\,;\, j\in\{1,\dots,m\}\rangle_{\mathbf{k}[x,s,t]}.$$

□

The next lemma determines the sub-algebra $\mathbf{k}[x,s,t][\langle x^N\rangle^{\mathbf{c}}/x^N]$ for every $N\in\mathbb{N}$.

Lemma 4.8. *The sub-algebra $\mathbf{k}[x,s,t][\langle x^N\rangle^{\mathbf{c}}/x^N]\subset B_{(n,e,Q)}$ is given by:*
(1) $\mathbf{k}[x,s,t][\langle x^{n_0}\rangle^{\mathbf{c}}/x^{n_0}] = \mathbf{k}[x,s,t,x^{n-n_0}y] = \mathbf{R}_{(n_0,S^d+T^r+XQ)}$ *for every* $n_0\in\{1,\dots,n-1\}$.
(2) $\mathbf{k}[x,s,t][\langle x^{n}\rangle^{\mathbf{c}}/x^{n}] = \cdots = \mathbf{k}[x,s,t][\langle x^{nm}\rangle^{\mathbf{c}}/x^{nm}] = \mathbf{k}[x,s,t,y] = \mathbf{R}_{(n,S^d+T^r+XQ)}$.
(3) $\mathbf{k}[x,s,t][\langle x^{nm+e_0}\rangle^{\mathbf{c}}/x^{nm+e_0}] = \mathbf{k}[x,s,t][y,x^{e-e_0}z] = B_{(n,e_0,Q)}$ *for every* $e_0\in\{1,\dots,e\}$.
Consequently, the exponential chain of $B_{(n,e,Q)}$ is

$$\mathbf{k}[x,s,t]\subset\mathbf{R}_{(1,S^d+T^r+XQ)}\subset\cdots\subset\mathbf{R}_{(n,S^d+T^r+XQ)}\subset B_{(n,1,Q)}\subset\cdots\subset B_{(n,e,Q)}$$

where $\mathbf{R}_{(n_0,S^d+T^r+XQ)}$ is the Russell $\mathbf{k}$-domain corresponding to the pair (n_0, S^d+T^r+XQ).

Proof. For (1), by Lemma 4.7, $\langle x^{n_0}\rangle^{\mathbf{c}} = \langle F,\, x^{n_0}\rangle_{\mathbf{k}[x,s,t]}$ for every $n_0\in\{1,\dots,n\}$. Therefore,

$$\mathbf{k}[x,s,t][\langle x^{n_0}\rangle^{\mathbf{c}}/x^{n_0}] = \mathbf{k}[x,s,t][F/x^{n_0}] = \mathbf{k}[x,s,t][x^{n-n_0}y] = \mathbf{R}_{(n_0,S^d+T^r+XQ)}.$$

For (2), it is enough to show that $\mathbf{k}[x,s,t][\langle x^{n}\rangle^{\mathbf{c}}/x^{n}] = \mathbf{k}[x,s,t][\langle x^{nm}\rangle^{\mathbf{c}}/x^{nm}]$. Lemma 4.7, asserts that $\langle x^{nm}\rangle^{\mathbf{c}} = \langle F^m, x^nF^{m-1},\dots,x^{(m-1)n}F, x^{nm}\rangle_{\mathbf{k}[x,s,t]}$. Therefore,

$$\mathbf{k}[x,s,t][\langle x^{nm}\rangle^{\mathbf{c}}/x^{nm}] = \mathbf{k}[x,s,t][F^m/x^{nm}, x^nF^{m-1}/x^{nm},\dots,x^{(m-1)n}F/x^{nm}].$$

Thus, we get

$$\mathbf{k}[x,s,t][\langle x^{nm}\rangle^{\mathbf{c}}/x^{nm}] = \mathbf{k}[x,s,t][y^m,\dots,y] = \mathbf{k}[x,s,t,y] = \mathbf{k}[x,s,t][I_n/x^n] = \mathbf{R}_{(n,S^d+T^r+XQ)}.$$

For (3), Lemma 4.7, assets that $I_{nm+e_0} = \langle G, \, x^{e_0}F^m, x^{n+e_0}F^{m-1}, \dots, x^{n(m-1)+e_0}F, x^{nm+e_0}\rangle_{\mathbf{k}[x,s,t]}$. Therefore,

$$\mathbf{k}[x,s,t][\langle x^{nm+e_0}\rangle^{\mathbf{c}}/x^{nm+e_0}] = \mathbf{k}[x,s,t][G/x^{nm+e_0}, \, x^{e_0}F^m/x^{nm+e_0}, \dots, x^{n(m-1)+e_0}F/x^{nm+e_0}].$$

Thus, we get

$$\mathbf{k}[x,s,t][\langle x^{nm+e_0}\rangle^{\mathbf{c}}/x^{nm+e_0}] = \mathbf{k}[x,s,t][G/x^{nm+e_0}, F/x^n] = \mathbf{k}[x,s,t][x^{e-e_0}z, y] = B_{(n,e_0,Q)}.$$

□

Remark 4.9. Lemma 4.7 and Lemma 4.8 show that the contraction chain $\mathbf{k}[x,s,t] \supset \langle x\rangle^{\mathbf{c}} \supset \langle x^2\rangle^{\mathbf{c}} \supset \cdots \supset \langle x^{nm+e}\rangle^{\mathbf{c}}$, associatrd to $B_{(n,e,Q)}$, consists of $nm+e$ distinct ideals in $\mathbf{k}[x,s,t]$, while the induced exponential chain $\mathbf{k}[x,s,t] \subsetneq \mathbf{R}_{(1,S^d+T^r+XQ)} \subsetneq \cdots \subsetneq \mathbf{R}_{(n,S^d+T^r+XQ)} \subsetneq B_{(n,1,Q)} \subsetneq \cdots \subsetneq B_{(n,e,Q)}$ has only $n+e$ distinct (even non-isomorphic by virtue of Proposition 4.1) sub-algebras. This will be a key observation to prove Proposition 4.11 and Theorem 5.3, that is, to distinguish $B_{(n,e,Q)}$; $e \neq 0$ from Russell domains. As we will see, the contraction chain of a Russell domain $\mathbf{R}_{(n',F)}$ consists of n' distinct ideals in $\mathbf{k}[x,s,t]$, and the exponential chain consists also of n' non-isomorphic sub-algebras. Therefore, in a sense, the number of non-isomorphic sub-algebras of the exponential chain, represents a numeric characterization for these $\mathbf{k}$-domains.

The following corollary is a consequence of Theorem 4.5 and Lemma 2.7.

Corollary 4.10. *The exponential chain* $\mathbf{k}[x,s,t] \subset \mathbf{k}[x,s,t][\langle x\rangle^{\mathbf{c}}/x] \subset \cdots \subset \mathbf{k}[x,s,t][\langle x^{nm+e}\rangle^{\mathbf{c}}/x^{nm+e}] = B_{(n,e,Q)}$ *of* $B_{(n,e,Q)}$ *is invariant by every locally nilpotent derivation of* $B_{(n,e,Q)}$. *That is, every* $\partial \in \mathrm{LND}(B_{(n,e,Q)})$ *restricts to a locally nilpotent derivation of every member of the exponential chain*

4.4. **Isomorphism classes and Automorphism groups.**

In the following proposition we give the necessary conditions that $B_{(n_1,e_1,Q_1)}$ and $B_{(n_2,e_2,Q_2)}$, where $n_1 + e_1 = n_2 + e_2$, must satisfy to be isomorphic. This will be done by comparing their exponential chains $\mathbf{k}[x,s,t] \subsetneq \mathbf{R}_{(1,S^d+T^r+XQ_1)} \subsetneq \cdots \subsetneq \mathbf{R}_{(n_1,S^d+T^r+XQ_1)} \subsetneq B_{(n_1,1,Q_1)} \subsetneq \cdots \subsetneq B_{(n_1,e_1,Q_1)}$ and $\mathbf{k}[x,s,t] \subsetneq \mathbf{R}_{(1,S^d+T^r+XQ_2)} \subsetneq \cdots \subsetneq \mathbf{R}_{(n_2,S^d+T^r+XQ_2)} \subsetneq B_{(n_2,1,Q_2)} \subsetneq \cdots \subsetneq B_{(n_2,e_2,Q_2)}$.

Proposition 4.11. *Suppose that* $B_{(n_1,e_1,Q_1)} \simeq B_{(n_2,e_2,Q_2)}$, *then* $n_1 = n_2$, *and* $e_1 = e_2$.

Proof. Let $\Psi : B_{(n_1,e_1,Q_1)} \longrightarrow B_{(n_2,e_2,Q_2)}$ be a $\mathbf{k}$-isomorphism, and assume for contradiction that $n_1 < n_2$. By Theorem 4.5 and Lemma 4.8, Ψ restricts to a $\mathbf{k}$-isomorphism between $\mathbf{k}[x,s,t][\langle x^{n_1}\rangle^{\mathbf{c}}_{B_{(n_1,e_1,Q_1)}}/x^{n_1}] = \mathbf{R}_{(n_1,S^d+T^r+XQ_1)}$ and $\mathbf{k}[x,s,t][\langle x^{n_1}\rangle^{\mathbf{c}}_{B_{(n_2,e_2,Q_2)}}/x^{n_1}] = \mathbf{R}_{(n_1,S^d+T^r+XQ_2)}$.
Consider the sub-algebra $\mathbf{k}[x,s,t][\langle x^{n_1+1}\rangle^{\mathbf{c}}_{B_{(n_1,e_1,Q_1)}}/x^{n_1+1}]$, it coincides with $\mathbf{R}_{(n_1,S^d+T^r+XQ_1)}$ by virtue of Lemma 4.8. On the other hand, by Theorem 4.5 $\mathbf{k}[x,s,t][\langle x^{n_1+1}\rangle^{\mathbf{c}}_{B_{(n_1,e_1,Q_1)}}/x^{n_1+1}]$ is isomorphic to $\mathbf{k}[x,s,t][\langle x^{n_1+1}\rangle^{\mathbf{c}}_{B_{(n_2,e_2,Q_2)}}/x^{n_1+1}]$ which coincides with $\mathbf{R}_{(n_1+1,S^d+T^r+XQ_2)}$. However, the latter is not isomorphic to $\mathbf{R}_{(n_1,S^d+T^r+XQ_2)}$ by virtue of Corollary 4.2, a contradiction. Thus $n_1 \geq n_2$ and by symmetry we deduce that $n = n_1 = n_2$. Since $n_1 + e_1 = n_2 + e_2$ by virtue of Proposition 4.1, we get $e = e_1 = e_2$, and we are done. □

Denote by $\mathrm{Iso}_{\mathbf{k}}\left(B_{(n_1,e_1,Q_1)}, B_{(n_2,e_2,Q_2)}\right)$ the set of all $\mathbf{k}$-isomorphisms from $B_{(n_1,e_1,Q_1)}$ to $B_{(n_2,e_2,Q_2)}$. Proposition 4.11 implies that this set is empty whenever $(n_1,e_1) \neq (n_2,e_2)$. The next proposition describes the set $\mathrm{Iso}_{\mathbf{k}}\left(B_{(n,e,Q_1)}, B_{(n,e,Q_2)}\right)$ in terms of a sub-set of $\mathrm{Aut}_{\mathbf{k}}(\mathbf{k}[x,s,t])$ (the group of $\mathbf{k}$-automorphisms of $\mathbf{k}[x,s,t]$). Let $\mathcal{A}$ be the sub-set of $\mathrm{Aut}_{\mathbf{k}}(\mathbf{k}[x,s,t])$ of automorphisms which preserve the ideal $\langle x\rangle_{\mathbf{k}[x,s,t]}$ and map $I = \langle x^{nm+e}\rangle^{\mathbf{c}}_{B_{(n,e,Q_1)}}$ isomorphically to $J = \langle x^{nm+e}\rangle^{\mathbf{c}}_{B_{(n,e,Q_2)}}$, that is,

$$\mathcal{A} := \{\psi \in \mathrm{Aut}_{\mathbf{k}}(\mathbf{k}[x,s,t]);\ \psi(x) = \lambda x\,;\ \lambda \in \mathbf{k}\setminus\{0\},\ \psi(I) = J\}.$$

Then,

Theorem 4.12. *There is a one-to-one correspondence between the set* $\mathrm{Iso}_{\mathbf{k}}\left(B_{(n,e,Q_1)}, B_{(n,e,Q_2)}\right)$ *and the set of* $\mathbf{k}$*-automorphisms* $\mathcal{A}$.

Proof. Every $\mathbf{k}$-isomorphisms $\Psi : B_{(n,e,Q_1)} \longrightarrow B_{(n,e,Q_2)}$ restricts to $\Psi|_{\mathbf{k}[x,s,t]}$ a $\mathbf{k}$-automorphism of the Derksen invariant $\mathbf{k}[x,s,t]$. On the other hand, Proposition 4.1 and Theorem 4.5 ensure that Ψ preserves the ideal $\langle x\rangle_{\mathbf{k}[x,s,t]}$ and that $\Psi(I) = J$. Conversely, every $\mathbf{k}$-automorphism ψ of $\mathbf{k}[x,s,t]$ that preserves the ideal $\langle x\rangle_{\mathbf{k}[x,s,t]}$ and satisfies $\psi(I) = J$ extends, by virtue of Lemma 3.2, in a unique way to $\widetilde{\psi}$ a $\mathbf{k}$-isomorphism between $\mathbf{k}[x,s,t][I/x^{nm+e}]$ and $\mathbf{k}[x,s,t][J/x^{nm+e}]$. These rings coincide with $B_{(n,e,Q_1)}$ and $B_{(n,e,Q_2)}$ by virtue of Lemma 4.8. And we are done. $\square$

The next corollary is a direct consequence of Theorem 4.12. It describes the $\mathbf{k}$-automorphism group of $B_{(n,e,Q)}$ as a sub-group of the $\mathbf{k}$-automorphism group of the Derksen invariant $\mathbf{k}[x,s,t]$.

Corollary 4.13. *The group* $\mathrm{Aut}_{\mathbf{k}}(B_{(n,e,Q)})$ *is isomorphic to the group* $\mathcal{A}$ *via the group isomorphism:*

$$\urcorner : \mathrm{Aut}_{\mathbf{k}}(B_{(n,e,Q)}) \xrightarrow{\sim} \mathcal{A}\ ;\ \ \urcorner(\Psi) = \Psi|_{\mathbf{k}[x,s,t]}$$

where $\Psi|_{\mathbf{k}[x,s,t]}$ *is the restriction of* $\Psi \in \mathrm{Aut}_{\mathbf{k}}(B_{(n,e,Q)})$ *to the sub-algebra* $\mathbf{k}[x,s,t] \subset B_{(n,e,Q)}$.

Consider the exponential chain of $B_{(n,e,Q)}$ [1]

$$\mathbf{k}[x,s,t] \hookrightarrow \mathbf{R}_{(1,S^d+T^r+XQ)} \hookrightarrow \cdots \hookrightarrow \mathbf{R}_{(n,S^d+T^r+XQ)} \hookrightarrow B_{(n,1,Q)} \hookrightarrow \cdots \hookrightarrow B_{(n,e,Q)}.$$

Every member of this chain represents an invariant sub-algebra of $B_{(n,e,Q)}$, and we have the following.

$$\begin{array}{ccccc} & & \mathrm{Aut}_{\mathbf{k}}(\mathbf{R}_{(1,S^d+T^r+XQ)}) & & \\ & & \cup & & \\ \mathrm{Aut}_{\mathbf{k}}(B_{(n,e,Q)}) \subset \cdots \subset \mathrm{Aut}_{\mathbf{k}}(B_{(n,1,Q)}) \subset \mathrm{Aut}_{\mathbf{k}}(\mathbf{R}_{(n,S^d+T^r+XQ)}) \subset \cdots \subset & & \mathrm{Aut}_{\mathbf{k}}(\mathbf{R}_{(2,S^d+T^r+XQ)}) & \subset & \mathrm{Aut}_{\mathbf{k}}(\mathbf{k}[x,s,t]), \end{array}$$

$\mathrm{LND}(B_{(n,e,Q)}) = x\left(\mathrm{LND}(B_{(n,e-1,Q)})\right) = \cdots = x^e\left(\mathrm{LND}(\mathbf{R}_{(n,S^d+T^r+XQ)})\right) = x^{n+e}\left(\mathrm{LND}_{\mathbf{k}[x]}(\mathbf{k}[x,s,t])\right)$ and

$\mathrm{LND}(\mathbf{R}_{(n,S^d+T^r+XQ)}) = x\left(\mathrm{LND}(\mathbf{R}_{(n-1,S^d+T^r+XQ)})\right) = \cdots = x^{n-2}\left(\mathrm{LND}(\mathbf{R}_{(2,S^d+T^r+XQ)})\right) = x^n\left(\mathrm{LND}_{\mathbf{k}[x]}(\mathbf{k}[x,s,t])\right)$.

5. New Exotic Structures on $\mathbb{C}^3$

Let $m,d,r \geq 2$ be fixed such that $\gcd(d,r)=1$. For every $e \geq 0$, $n \geq 1$ such that $(n,e) \neq (1,0)$, and every $Q \in \mathbf{k}[X,S,T]$, we denote by $B_{(n,e,Q)}$ the following $\mathbf{k}$-domain:

$$B_{(n,e,Q)} := \mathbf{k}[x,y,z,s,t] \simeq \mathbf{k}[X,Y,Z,S,T]/\langle X^nY - S^d - T^r - X\,Q(X,S,T),\ Y^m - X^eZ - S\rangle.$$

Definition 5.1. Recall that a smooth affine variety which is diffeomorphic to $\mathbb{R}^{2N}$ but not isomorphic to $\mathbb{C}^N$ is called an *exotic* $\mathbb{C}^N$.

5.1. A class of exotic threefolds.

Let $\mathbf{k}=\mathbb{C}$ and assume that $Q(0,0,0) \neq 0$ and $e \gneq 1$, then, by the Jacobian criterion, the variety $V' = \mathrm{Sped}(B_{(n,e,Q)})$ is the smooth threefold $x^ny - (y^m - x^ez)^d - t^r - xQ$ in $\mathbb{C}^4$, which birationally dominates the affine space $V = \mathbb{C}^3$ under the blowup morphism $\sigma_I : V' \longrightarrow V = \mathbb{C}^3$; $\sigma_I(x,y,z,t) \mapsto (x, y^m - x^ez, t)$. The exceptional divisor of the affine modification $\sigma_I : V' \longrightarrow V$, see Proposition 4.4, coincides with $\mathrm{Spec}(A) := \{x=0\} \subset V'$ where $A := \mathbb{C}[s,t,y,z] \simeq \mathbb{C}[S,T,Y,Z]/\langle S^d+T^r, Y^m - S\rangle \simeq \mathbb{C}[T,Y,Z]/\langle Y^{md}+T^r\rangle$, hence $\mathrm{Spec}(A) \simeq \mathbb{C} \times \Gamma_{md,r}$ where $\Gamma_{md,r} = \mathrm{Spec}(\mathbb{C}[Y,T]/\langle Y^{md}+T^r\rangle)$. Assume in addition that $\gcd(m,r)=1$. Since every irreducible singular curve of the form $\Gamma_{N_1,N_2} = \mathrm{Spec}(\mathbb{C}[Y,T]/\langle Y^{N_1}+T^{N_2}\rangle)$ where $\gcd(N_1,N_2)=1$, $N_1 > N_2 \geq 2$, is contractible, see [16]. We conclude that the necessary conditions, see [21, Proposition 4.2], for preserving the topology under affine modifications are fulfilled. Therefore, by [21, Theorem 4.3], the variety V' is contractible as a complex threefold, which yields that V' is diffeomorphic to $\mathbb{R}^6$ by virtue of the Dimca-Ramanujam Theorem [21, Theorem 3.2]. Since $B_{(n,e,Q)}$ is not isomorphic to $\mathbb{C}^{[3]}$ by virtue of Theorem 2.6 or Corollary 2.9, we deduce that V' is not isomorphic to the affine space $\mathbb{C}^3$. Therefore, $V' = \mathrm{Spec}(B_{(n,e,Q)})$ is an exotic $\mathbb{A}^3_{\mathbb{C}}$.

Note that since $B_{(n,e,Q)}/\langle x\rangle \simeq \mathbf{k}[Y,Z,S,T]/\langle S^d+T^r, Y^m - S\rangle \simeq \mathbf{k}[Y,Z,T]/\langle Y^{md}+T^r\rangle$, the principle ideal $\langle x\rangle$ is prime whenever $\gcd(m,r)=1$. On the other hand, $B_{(n,e,Q)}[x^{-1}] = \mathbf{k}[x^{-1},x,s,t]$ the localization of B with respect to x, is a unique fraction domain, therefore $B_{(n,e,Q)}$ is also a unique fraction domain by virtue of [20, Lemma 1].

We put together the previous observations in the following.

Theorem 5.2. *Under the conditions:* ($\mathbf{k}=\mathbb{C}$, $\gcd(m,r)=1$, $e \geq 2$, *and* $Q(0,0,0) \neq 0$). *The smooth factorial variety* $\mathrm{Spec}(B_{(n,e,Q)})$ *is diffeomorphic to* $\mathbb{R}^6$ *but not isomorphic to* $\mathbb{C}^3$. *Hence,* $\mathrm{Spec}(B_{(n,e,Q)})$ *is an exotic* $\mathbb{C}^3$.

[1] Particular members of the exponential chain of $B_{(n,e,Q)}$ are $\mathbf{k}[x,s,t] \hookrightarrow R_{(n,S^d+T^r+XQ)} \hookrightarrow B_{(n,e,Q)}$. They correspond to $\mathrm{AL}_0(B_{(n,e,Q)}) \hookrightarrow \mathrm{AL}_\alpha(B_{(n,e,Q)}) \hookrightarrow \mathrm{AL}_{m\alpha}(B_{(n,e,Q)})$ for $\alpha = \min\{d,r\}$, see [1, Section 2] for definitions and some properties of $\mathrm{AL}_{i\in\mathbb{N}}$-invariants.

5.2. **Comparing the class $B_{(n,e,Q)}$ with Russell domains.**
Here, we prove that domains of the form $B_{(n,e,Q)}$; $e \neq 0$ are not isomorphic to any of Russell $\mathbf{k}$-domains.

Denote by $\mathbf{R}_{(n',F)}$ the Russell $\mathbf{k}$-domain corresponding to the pair (n', F), that is,

$$\mathbf{R}_{(n',F)} := \mathbf{k}[x,s,t,y] \simeq \mathbf{k}[X,Y,S,T]/\langle X^{n'}Y - F(X,S,T)\rangle.$$

Theorem 5.3. *Suppose that $B_{(n,e,Q)} \simeq \mathbf{R}_{(n',F)}$, then $e = 0$ and $n = n'$.*

Proof. Suppose that $B_{(n,e,Q)} \simeq \mathbf{R}_{(n',F)}$, then both rings have the same Derksen and Makar-Limanov invariants. Therefore, by Theorem 2.6 and Corollary 2.9, the Derksen and Makar-Limanov invariant of $\mathbf{R}_{(n',F)}$ is $\mathbf{k}[x,s,t]$ and $\mathbf{k}[x]$ respectively, where we realize both $\mathbf{k}$-domains as sub-algebras of $B_{(n,e,Q)}[x^{-1}] = \mathbf{R}_{(n',F)}[x^{-1}] = \mathbf{k}[x^{-1},x,s,t]$.
Let $\Psi : B_{(n,e,Q)} \longrightarrow \mathbf{R}_{(n',F)}$ be a $\mathbf{k}$-isomorphism between $B_{(n,e,Q)}$ and $\mathbf{R}_{(n',F)}$, then it restricts to a $\mathbf{k}$-automorphism of the Makar-Limanov invariant $\mathbf{k}[x]$. Hence, $\Psi(x) = \lambda x + c$ for some $\lambda \in \mathbf{k}\backslash\{0\}$ and $c \in \mathbf{k}$. Therefore, Ψ induces $\overline{\Psi}$ an isomorphism between $B_{(n,e,Q)}/\langle x\rangle$ and $\mathbf{R}_{(n',F)}/\langle \lambda x + c\rangle$, which implies that $c = 0$. Indeed, assume that $c \neq 0$, then $\mathbf{R}_{(n',F)}/\langle \lambda x + c\rangle \simeq \mathbf{k}[S,T] \simeq \mathbf{k}^{[2]}$. On the other hand, $B_{(n,e,Q)}/\langle x\rangle \simeq \mathbf{k}[Y,T,Z]/\langle Y^{md} + T^r\rangle$ is either a non-domain (if $\gcd(m,r) \neq 1$) or a semi-rigid $\mathbf{k}$-domain with ML-invariant equal to $\mathbf{k}[Y,T]/\langle Y^{md} - T^r\rangle$, see [18, Lemma 21]. Either way $B_{(n,e,Q)}/\langle x\rangle$ is not isomorphic to $\mathbf{k}^{[2]}$ and hence the only possibility for c is that $c = 0$. Thus we have $\Psi(x) = \lambda x$. Furthermore, since $\mathbf{R}_{(n',F)}/\langle x\rangle \simeq \mathbf{k}[S,T,Y]/P(S,T)$ where $P(S,T) := F(0,S,T)$, we can assume that $P(S,T) = S^{md} + T^r$. Observe that $\mathbf{R}_{(n',F)}$ is the exponential modification of $\mathbf{k}[x,s,t]$ with locos $(x^{n'}, \langle x^{n'}, F\rangle_{\mathbf{k}[x,s,t]})$ and exponential chain $\mathbf{k}[x,s,t] \subset \mathbf{R}_{(1,F)} \subset \cdots \subset \mathbf{R}_{(n',F)}$.
The same argument, as in the proof of Proposition 4.1 or Theorem 3.5, shows that $n + e = n'$. Assume for contradiction that $e \neq 0$, then $n < n'$. Theorem 3.5 asserts that Ψ maps the contraction of $\langle x^N\rangle_{B_{(n,e,Q)}}$ isomorphically to the contraction of $\langle x^N\rangle_{\mathbf{R}_{(n',F)}}$, and that Ψ maps the sub-algebra $\mathbf{k}[x,s,t][\langle x^N\rangle^{\mathbf{c}}_{B_{(n,e,Q)}}/x^N] \subset B_{(n,e,Q)}$ isomorphically to the sub-algebra $\mathbf{k}[x,s,t][\langle x^N\rangle^{\mathbf{c}}_{\mathbf{R}_{(n',F)}}/x^N] \subset \mathbf{R}_{(n',F)}$, for every $N \in \mathbb{N}$. In particular, Ψ restricts to a $\mathbf{k}$-isomorphism between $\mathbf{R}_{(n,S^d+T^r+XQ)} = \mathbf{k}[x,s,t][\langle x^n\rangle^{\mathbf{c}}_{B_{(n,e,Q)}}/x^n] \subset B_{(n,e,Q)}$ and $\mathbf{k}[x,s,t][\langle x^n\rangle^{\mathbf{c}}_{\mathbf{R}_{(n',F)}}/x^n] = \mathbf{R}_{(n,F)} \subset \mathbf{R}_{(n',F)}$. Consider the sub-algebra $\mathbf{k}[x,s,t][\langle x^{n+1}\rangle^{\mathbf{c}}_{B_{(n,e,Q)}}/x^{n+1}] \subset B_{(n,e,Q)}$, it is equal to $\mathbf{R}_{(n,S^d+T^r+XQ)}$ by virtue of Lemma 4.8. On the other hand, the sub-algebra $\mathbf{k}[x,s,t][\langle x^{n+1}\rangle^{\mathbf{c}}_{\mathbf{R}_{(n',F)}}/x^{n+1}] = \mathbf{R}_{(n+1,F)}$ is not equal (even non-isomorphic) to $\mathbf{R}_{(n,F)}$, a contradiction. Therefore, the only possibility is $e = 0$ and $n = n'$, as desired. ∎

As a consequence of Theorem 5.3, we have the following.

Corollary 5.4. *Under the conditions:* ($\mathbf{k} = \mathbb{C}$, $\gcd(m,r) = 1$, $e \geq 2$, *and* $Q(0,0,0) \neq 0$). *The variety* $\mathrm{Spec}(B_{(n,e,Q)})$ *is not isomorphic to* $\mathrm{Spec}(\mathbf{R}_{(n',F)})$. *Consequently,* $\mathrm{Spec}(B_{(n,e,Q)})$ *represents a new exotic* $\mathbb{C}^3$.

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